# NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

## TECHNICAL NOTE 2977

## TECHNIQUES FOR CALCULATING PARAMETERS OF NONLINEAR DYNAMIC SYSTEMS FROM RESPONSE DATA

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## SUMMARY

A study has been made of the problem of determining nonlinear parameters of dynamic systems, such as aircraft or servomechanisms, whose measured responses are nonlinear. This problem cannot be solved in general but, for certain types of systems, adequate answers can be obtained. In the differential equations representing the motion of the latter systems, one or more coefficients will be functions of the nonlinear parameters. For the systems studied herein, the nonlinear parameters are functions of either the amplitude of the dependent variables or their time derivatives. The techniques developed for coefficient evaluation are, for the most part, extensions of two well-known methods of stability derivative evaluation for linear systems, namely, the derivative method and the method of inspection of transients for period and damping characteristics. Phase-plane methods such as those utilized in nonlinear mechanics are also included.

It was found that the nonconstant coefficients could be determined satisfactorily for first- or second-order systems, using one or a combination of the techniques developed. For higher-order systems the only practical means of obtaining results was found to be the derivative method. For some of the examples tried, however, this latter method gave poor results due to the occurrence of ill-conditioned algebraic equations. The number of unknown coefficients, constant or nonconstant, to be determined also affected the accuracy of the results, the smaller the number, of course, the better the chances for a satisfactory evaluation.

## INTRODUCTION

Much attention has been given recently to the problem of determining stability derivatives of airplanes by means of flight and wind-tunnel dynamic testing. This problem of reducing response data to obtain stability parameters is the inverse problem in dynamics, the direct problem being to find the response to an arbitrary forcing function when the

equations of motion and the system parameters are known. For linear systems, methods for solving the inverse problem are summarized and evaluated in reference 1. More detailed descriptions of the individual methods can be found in references 2 through 6.

In the investigation reported herein, the problem of determining nonlinear stability parameters from dynamic responses of nonlinear systems was studied. Previous to performing the actual analytical operations involved in this inverse type of analysis, it is desirable to determine the magnitude of the nonlinearities involved in order to classify the problem as either approximately linear or sufficiently nonlinear to necessitate the more complicated nonlinear approach. It is also necessary to make an assumption about the order and form of the differential equation of the system being investigated. The response of a system to sinusoidal inputs can be analyzed to find the magnitude of the nonlinearity involved, but there is no criterion for the identification of the order and form of a differential equation to represent the system. The latter must be determined more or less intuitively.

There is a variety of effects which may cause nonlinear behavior in a dynamic system. These effects appear in several different ways in the equations of motion of the system. The cause of one common nonlinear term is an element in the system whose response is dependent on the amplitude of a dependent variable. For example, it is possible for an airplane to have a nonlinear variation of pitching moment with angle of attack which would require a nonlinear term $C_m(\alpha)$ to appear in the equations of motion in place of the usual linear term $C_{m_\alpha}\alpha$. The techniques presented in this report are concerned essentially with systems having this type of nonlinearity.

This investigation was mainly concerned with aircraft responses. The techniques presented, however, are shown to be applicable to responses of other mechanisms such as an autopilot servo system. The degree of nonlinearity in the responses for high-speed aircraft of unorthodox configuration, such as a guided missile, is already a matter of concern to design engineers. (See ref. 7.) Furthermore, the automatic control systems employed in aircraft usually contain, within the servo system itself, nonlinearities due to factors such as amplifier saturation and velocity limitation.

For the most part, the methods developed herein are extensions of certain linear techniques. In reference 1 the several methods for analyzing linear responses are placed in two categories. The first category includes methods wherein the response data are substituted directly into the differential equations, the general forms of which are assumed. Such methods are called "equations-of-motion methods." Examples of techniques in this category are the "derivative method" given in references 2 and 3, and the "matrix method" of reference 4. The second category involves methods wherein analytical curves are fitted to the dynamic responses. These techniques are known as "response curve-fitting

methods" and the methods described in references 5 and 6 are examples of this type. The same categorical divisions seem suitable for the nonlinear techniques derived from these linear ones. In addition, a technique of analysis based on phase-plane methods, which does not fit precisely into either of the above two categories, is presented for use in determining nonlinear damping functions associated with certain classes of self-sustained oscillations.

Experience has shown that the results of the analysis of first- and second-order linear systems are more accurate than those of systems of higher order. It is not at all surprising, therefore, that this condition also exists in the nonlinear situation. Fortunately, in many cases the dynamic behavior of systems can be adequately described by second-order differential equations whether linear or nonlinear. Emphasis has therefore been placed on techniques suitable for second-order systems. The derivative method, however, is applicable in principle in cases where the differential equations involved are of any order whatever, the primary limitation being the accuracy in determining the higher-order derivatives of the response either by measurement or computation. In order to show that this limitation is not always serious, a successful demonstration of the technique of analysis for a nonlinear system of fourth order with unknown parameter is presented in the illustrative examples.

## NOTATION

$C_{m_\alpha}$ rate of change of pitching-moment coefficient with angle of attack, $\frac{\partial C_m}{\partial \alpha}$

$C_{Z_\alpha}$ rate of change of normal-force coefficient with angle of attack, $\frac{\partial C_Z}{\partial \alpha}$

$C_{Z_\delta}$ rate of change of normal-force coefficient with elevator deflection angle, $\frac{\partial C_Z}{\partial \delta}$

$C_{m_q}$ rate of change of pitching-moment coefficient with pitching velocity, $\frac{\partial C_m}{\partial q}$

$C_m(\alpha)$ nonlinear pitching-moment coefficient

$C_m{*}(\alpha)$ nonlinear pitching-moment parameter (See eq. (B15).)

$\frac{\partial C_m^*}{\partial \alpha}$ nonlinear undamped natural-frequency parameter (See eq. (B16).)

$F(x), F(\dot{x})$ nonlinear damping coefficients, arbitrary second-order system

$G(x)$ nonlinear restoring-force coefficient, arbitrary second-order system

$H(t)$ forcing function, arbitrary second-order system

$I_y$ moment of inertia about y axis

$I_z$ moment of inertia about z axis

$I(v_e)$ nonlinear amplifier characteristic, autopilot servo

K nondimensional moment-of-inertia parameter, $\frac{I_y \rho S}{m^2 c}$

N yawing moment

$N_\psi$ rate of change of yawing moment with angle of yaw, $\frac{\partial N}{\partial \psi}$

$N_\beta$ rate of change of yawing moment with angle of sideslip, $\frac{\partial N}{\partial \beta}$

P period of oscillation

$P_a A_a$ slope of nonlinear amplifier characteristic, autopilot servo

$P_f$ gain parameter, autopilot servo

R radius of phase-plane limit cycles of Van der Pol and Rayleigh equations, small values of $\mu$

S wing area

$V_o$ free-stream velocity

b damping coefficient, second-order linear differential equation, $2\zeta\omega_n$

$\bar{c}$ local wing chord

c mean aerodynamic chord, $\frac{1}{S} \int_{-s}^{s} \bar{c}^2 \, d\bar{y}$

$f(x, \dot{x})$ nonlinear damping term, second-order nonlinear differential equation

| | |
|---|---|
| $k$ | restoring-force coefficient, second-order linear differential equation, $\omega_n^2$ |
| $k_f,k_m$ | parameters, autopilot servo |
| $m$ | mass |
| $q$ | pitching velocity |
| $t$ | time |
| $t_1$ | $\omega_n t$ |
| $t_{1/2}$ | time to damp to half amplitude |
| $v_f,v_i,v_e$ | output, input, and error voltages of autopilot servo |
| $x,y,z$ | arbitrary dependent variables |
| $\bar{y}$ | spanwise distance from plane of symmetry |
| $\alpha$ | angle of attack |
| $\beta$ | angle of sideslip |
| $\delta$ | elevator deflection angle |
| $\epsilon$ | parameter, Whittaker's smoothing formula |
| $\rho$ | density of air |
| $\tau$ | aerodynamic time, $\frac{m}{\rho S V_o}$ |
| $\tau_m,\tau_D$ | parameters, autopilot servo |
| $\psi$ | angle of yaw |
| $\mu$ | parameter, Van der Pol and Rayleigh equations |
| $\Delta,\Delta^2,\ldots$ | first, second, etc., differences |
| $\omega$ | angular frequency of oscillation, radians/sec |
| $\omega_n$ | undamped natural frequency |
| $\zeta$ | damping ratio |

Subscript

s smoothed by Whittaker's method

## SCOPE OF THE ANALYSIS

The dynamic system under consideration, with one exception, will be assumed to be describable by a second-order differential equation such as

$$\ddot{x} + f(x,\dot{x}) + g(x) = H(t) \tag{1}$$

where $f(x,\dot{x})$ is either of the form $F(x)\dot{x}$, or $F(\dot{x})\dot{x}$, or by one or two first-order equations of the form

$$\left.\begin{aligned} \dot{x} &= f_1(x,y,t) \\ \dot{y} &= f_2(x,y,t) \end{aligned}\right\} \tag{2}$$

In equations (1) and (2), x and y are arbitrary variables and the dots over x and y indicate differentiation with respect to time t. It will also be assumed that any nonlinear coefficients or parameters are functions only of the dependent variable or its first derivative, and not of the time.

The exceptional case included will be a fourth-order system. This example is presented to demonstrate the application of the derivative method to nonlinear systems of higher order. The dynamic system in this example is an autopilot servo, and the differential equation describing its open-loop behavior is assumed to be of the form

$$A_0 \frac{d^4 v_f}{dt^4} + A_1 \frac{d^3 v_f}{dt^3} + A_2 \frac{d^2 v_f}{dt^2} + A_3 \frac{dv_f}{dt} - I(v_e) = 0 \tag{3}$$

where $v_f$ and $v_e$ represent output and error voltages and the coefficients $A_0, A_1, A_2$, and $A_3$ are constants. The function $I(v_e)$ is nonlinear.

In order to facilitate the actual numerical work involved in the illustrative examples presented in the appendixes, a Reeves Electronic Analogue Computer and IBM digital computing equipment were used. For the purpose of obtaining harmonic analyses of certain responses a harmonic analyzer was also employed.

## DETERMINATION OF DEGREE OF NONLINEARITY

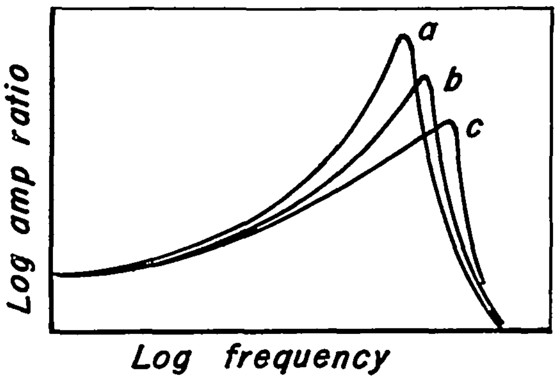
The degree of nonlinearity of a dynamic system, as determined by examination of response data, is measured by the deviation of the responses from those of a purely linear system. It can be considered as an indication of the magnitude of the physical nonlinearity present in a given system. For practical purposes it would be desirable to have some number, or figure of merit, which would be a quantitative measure of the intensity or degree of the nonlinearity. For example, a complex dynamic system may have a response that to all appearances is linear. However, if an analysis of the response data is to be attempted, it might save time and work if tests could be made to ascertain whether the data warranted linear or nonlinear treatment. All physical systems are essentially nonlinear to some extent, but if the degree of nonlinearity is small it is possible that linear analyses would suffice.

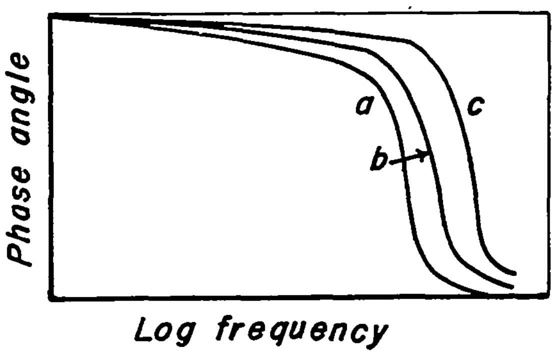
There are certain response characteristics which are typical of nonlinear systems and which may be recognized without resorting to a figure of merit. One example is a system which possesses a self-sustained oscillation. As another example, the output of the system to two sinusoidal inputs of different frequency may possess not only the two impressed frequencies, but also multiples, and sums and differences of multiples of the impressed frequencies. These nonlinear phenomena and many more are discussed in detail in references 8 and 9. These effects are readily recognized in dynamic responses and they cannot in general be described by linear differential equations. When they are observed, then, there is no question but that nonlinear analyses must be used.

On the other hand, there are cases where the effect of a nonlinearity in a dynamic system is not strongly evident upon cursory examination of a response. A transient may have appearances of a linear response, for example, until it is observed that the period of the oscillation changes as the oscillation damps, which could be the effect of a nonlinear restoring force. The time to damp to half amplitude may be observed to vary for transient oscillations about different trim positions, which is evidence of a nonlinearity in the damping.

The presence or absence of nonlinearities in a dynamic system can usually be established by the examination of the ratio of output to input amplitude for various sinusoidal input magnitudes at a given frequency. If the amplitude ratio or the phase angle shows a substantial variation, the system is effectively nonlinear. Such effects are most emphatically noted on frequency-response plots. These plots are commonly presented with the logarithm of the amplitude ratio and the phase angle as ordinates and the logarithm of the frequency as the abscissa. (See, e.g., ref. 10.) For a purely linear system these curves do not vary with the amplitude of the input. In nonlinear systems, however, the resonance frequency may shift and peak amplitudes may increase or

decrease as the input amplitude varies. In the accompanying sketch logarithmic plots for a nonlinear system for three different amplitudes of a sinusoidal input are shown. Any one of these curves has the appearance of a linear response but since the characteristics of the system change as the amplitude of the input changes it must be concluded that the system is nonlinear.




The deviations from linearity may be such that for engineering purposes, linear analyses could be used with sufficient accuracy. The simple techniques discussed in the preceding paragraphs, however, do not provide any figure of merit for so judging. If limits are set for tolerable variation of such quantities as time to damp to half amplitude, resonance frequency, peak amplitude, and phase shift for the range of inputs under consideration, then, it may be possible to establish whether or not linear analyses can be used by examining transient responses or frequency-response data.

As previously mentioned, however, it would be desirable to have a number or figure of merit indicative of the degree or intensity of nonlinearity for a given system. It is possible to make a quantitative estimate of this sort, if steady-state responses to sinusoidal inputs are obtained. The figure of merit, called the distortion factor,[1] is defined as 100 times the square root of the sum of the squares of the amplitudes of all harmonics present in the response beyond the fundamental, divided by the amplitude of the fundamental. For perfect sinusoidal response data obtained from a purely linear system, the distortion factor is zero. Data obtained from sinusoidal inputs to nonlinear systems, however, contain various amounts of higher harmonics, and the distortion factor is a quantitative measure of the amplitudes of these harmonics generated by the nonlinearities.

In Appendix A a demonstration of the calculation of the distortion factor is presented. This quantitative test of nonlinearity was applied to data used subsequently in the examples illustrating the application of data reduction methods for nonlinear systems. The results indicated that a distortion factor in excess of about 5 percent would probably warrant application of nonlinear analyses.

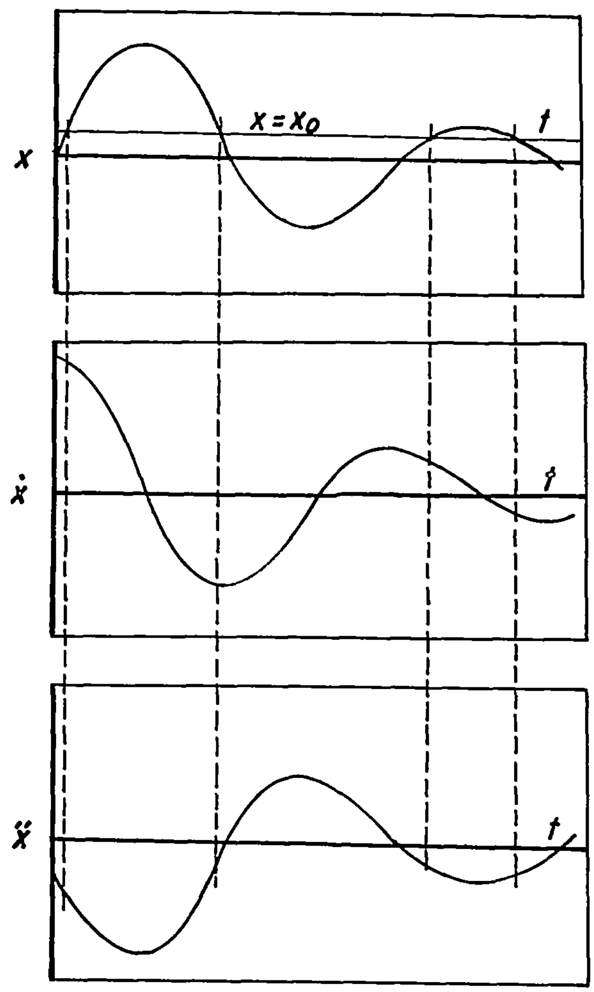
[1]The distortion factor is commonly used in electrical and communications engineering to measure wave distortion.

## REDUCTION OF DATA TO NONLINEAR PARAMETERS

### Equations-of-Motion Method

In reference 3 the derivative method, the only equations-of-motion technique that will be applied herein to nonlinear problems, is described for application to linear systems. This technique consists of obtaining the dynamic response of the system to a prescribed input, obtaining the proper number of response derivatives, and then for several instants of time substituting the corresponding values of input, response, and response derivatives into the differential equations of the system. In this manner a set of linear algebraic equations is developed wherein the variables are the coefficients of the differential equations. The coefficients are then calculated by solution of this set of algebraic equations either simultaneously or by least-squares methods. For linear dynamic systems a single response provides sufficient data for this calculation. A major difficulty in using this method of data reduction is in obtaining accurate derivatives of the output when they cannot be measured experimentally.

The derivative method can be applied to reduction of nonlinear data in the following way. Suppose that several oscillations of the response of the system to a given input are recorded. For the sake of simplifying the following discussion, let the given response be a free transient and assume that the system under consideration is of second order, describable by an equation of the general form of equation (1), where $f(x,\dot{x})$ is of the form $F(x)\dot{x}$. Obtain the first two derivatives of the response, then plot response and derivatives to the same time scale, as in the accompanying sketch. At a value of $x$, say $x_o$, draw a line parallel to the time axis on the $x$ time history. At each instant where this line cuts the response curve read the values of $\dot{x}$ and $\ddot{x}$. In this way several sets of values of $\dot{x}$ and $\ddot{x}$, all corresponding to the same value of $x$, namely $x_o$, are obtained. It should be pointed out here that there must be at least as many sets



of values of $\dot{x}$ and $\ddot{x}$ as there are unknown coefficients in the differential equations of the system.

Assume that the equation of motion of the system is of the form

$$\ddot{x} + F(x)\dot{x} + G(x)x = 0 \tag{4}$$

Also assume that for a given value of $x$, damping and restoring force coefficients have unique values. These unique values will be taken as the numbers which are found by solving by least squares the set of equations which are obtained by substituting the sets of values of $\dot{x}$ and $\ddot{x}$ corresponding to $x_o$, as determined by the graphical procedure of the previous paragraph, into equation (4). If this procedure is repeated for several more values of $x$, say $x_1$, $x_2$, etc., then the numbers $F(x_o)$, $F(x_1)$, etc., and $G(x_o)$, $G(x_1)$, etc., may be plotted against the corresponding values of $x$, and the nonlinear coefficients will then be represented graphically as functions of $x$.

Consider now the application of the derivative method in the case where only one coefficient, suspected to be nonconstant, is unknown. From a single response of this system the unknown coefficient can be calculated by the derivative method. The coefficient will be valid within the range of values of displacement as given by the experimental response curve. To compute the coefficient, merely substitute the response and its derivatives into the differential equation and solve directly for the unknown parameter as a function of time. These results are then plotted against displacement to give the nonlinear variation of the parameter with displacement.

Consider next the case where more than one coefficient is unknown. If it is assumed that all but one of these unknown coefficients are constant, there is a reasonable chance that the constant coefficients can be found by analyzing a small-amplitude response, using one of the linear techniques previously mentioned. However, if more than one coefficient must be solved for using the derivative method, there is a great possibility that the data used will result in ill-conditioned equations for the nonlinear systems. The apparent cause of this predicament is that all the values of the derivatives of the dependent variable must be read at the one value of the dependent variable for one set of simultaneous equations. As a result, all the derivatives tend to have nearly the same value in each equation. In order to avoid this difficulty it is desirable to use sets of data for which the particular value of the dependent variable being used occurs in widely separated parts of the response excursion. Such data could be obtained from sinusoidal inputs of different magnitudes or from a modulated sinusoidal input. A modulated input, wherein either the amplitude or frequency is modulated, has the advantage that the entire excursion range desired for the dependent variable could be covered in one input. The separate

sinusoidal inputs, however, were found to yield data that resulted in the less ill-conditioned simultaneous equations.

As a general rule in applying the derivative method it is well to determine as many of the constant coefficients as possible, beforehand, by other means. When the equations of motion of a system are derived there is one for each degree of freedom. It is preferable to use the several equations in the form in which they are derived, rather than to eliminate variables and obtain a single equation of high order. The necessity for taking derivatives of the output data will thereby be kept to a minimum.

In Appendix B several numerical examples are presented in order to demonstrate the derivative method for one or two unknown parameters. In the first of these examples the numerical techniques for smoothing and differentiating response data are used and discussed in some detail. Examples 1 and 2 show that for a single unknown parameter very good results can be obtained, even for a system of fourth order. For the case of more than one unknown, however, demonstrated in the third example, the results were of poor accuracy due to the occurrence of ill-conditioned equations in the computations. For second-order systems with more than one unknown, better results may be obtained by using the techniques presented in the next section.

## Response Curve-Fitting Methods

Curve-fitting methods for response-data reduction are based upon the assumption that the responses can in some manner be approximated, or fitted, by analytical curves whose characteristics are known. One of the most elementary of the linear curve-fitting techniques is to examine transient responses to determine the period and damping characteristics, from which a sum of exponentials which fits the given transient response can be written. A technique based upon this simple linear transient-response analysis method, applicable to second-order nonlinear data, will be presented here.[2]

To carry out this technique of data reduction, responses of the second-order system are obtained to step inputs of varying amplitude. Now, the assumption is made that small oscillations about each of the steady-state, or trim, displacements are linear, and a period and damping analysis is made of each response to determine period $P$, and time to damp to half amplitude $t_{1/2}$. If the oscillation is too strongly damped, of course, this calculation is not feasible. Second-order linear differential equations are commonly written in the form

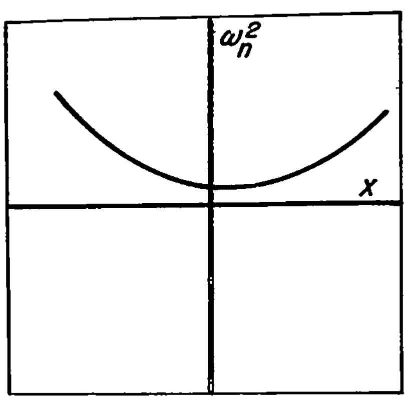
[2]The application of this well-known linear analysis method to nonlinear problems was suggested by Mr. Stewart M. Crandall of the Ames Aeronautical Laboratory of the NACA.

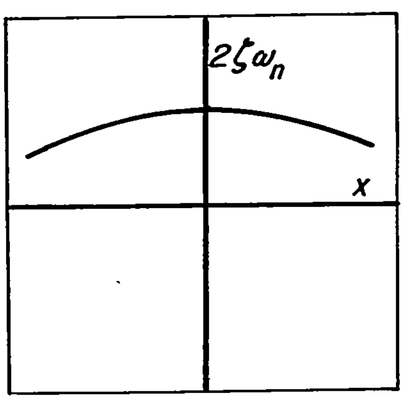
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = H(t) \tag{5}$$

(see, e.g., ref. 10) where $\zeta$ is the damping ratio and $\omega_n$ is the natural undamped frequency. It is also well known that when $H(t)$ is a step or pulse input, P and $t_{1/2}$ are related to $\zeta$ and $\omega_n$ in the following manner:

$$\left.\begin{aligned} t_{1/2} &= \frac{0.693}{\zeta\omega_n} \\ P &= \frac{2\pi}{\omega} \\ \omega &= \omega_n\sqrt{1 - \zeta^2} \end{aligned}\right\} \tag{6}$$

The values of P and $t_{1/2}$ for each of the step input responses are then substituted into equations (6) to determine the coefficients $2\zeta\omega_n$ and $\omega_n^2$, valid, under the present assumptions, at the particular trim value of the displacement.




The values of $\omega_n^2$ and $2\zeta\omega_n$ are then plotted against the trim displacement to which they correspond, thus determining their variation with displacement. The accompanying sketches illustrate this graphical procedure.

To demonstrate this technique, a numerical example is presented in Appendix B. The determination of both the damping coefficient and a restoring-force term of a second-order system by this method is found to be relatively easy and accurate.

A somewhat simpler technique which can be used only for finding restoring-force nonlinearities, based upon inspection of transient responses, can be developed from the steady-state method given in reference 3. For a step input to a second-order system, the response assumes a constant steady-state value, where the derivatives of

the response and the input are all zero. Thus, in the steady state, the differential equation is reduced to a relationship between the step input and the restoring force. By using this technique on the first system considered in Appendix B (see eq. (A3)), for example, $C_m(\alpha)$ can be determined from the following relationship:

$$C_m(\alpha_o) = \frac{1}{66.181} (2.285\alpha_o - 4.620\delta_o)$$

where $\alpha_o$ is the steady-state value of $\alpha$, resulting from a step input of magnitude $\delta_o$. In this manner, the nonlinear parameter can be determined as a function of steady-state values of the response, as was done previously in a more elaborate fashion. The advantage of this technique is that aperiodic responses can be analyzed, but a disadvantage is that the damping cannot be calculated directly.

Another curve-fitting method based on linear equivalence, so obvious that it is usually the first to be suggested, was tried. In this method, the linear damping or restoring-force equivalents obtained from excursions centered around zero displacement and covering various amplitudes of the response would be used to construct the nonlinear function. Unfortunately, however, no basis was found for translating these equivalent linear slopes into the correct nonlinear function. The simple technique of fairing through the end points of the resulting family of linear slopes did not provide a satisfactory result.

## Phase-Plane Methods, Self-Sustained Oscillations

In this section, an inverse process will be presented for the calculation of the nonlinear damping term in the differential equation of a second-order system which exhibits self-sustained oscillations. The snaking oscillations of an aircraft are self-sustained, for example, and it is possible that the nature of the nonlinear damping of such oscillations could be determined by phase-plane methods. It should be mentioned, however, that unless the damping corresponds fairly closely to the nonlinear damping of either the Van der Pol or Rayleigh classical equation of nonlinear mechanics, the phase-plane method presented here may not be useful.

For any oscillation having a periodic variation of the displacement with time, the time rate of change of displacement is likewise periodic. Thus, the phase-plane plot, displacement versus velocity, for such an oscillation is a closed curve. Moreover, if the closed curve is unique, that is, if the trajectory of the motion in the phase plane eventually reaches the closed curve regardless of initial conditions, the oscillation has a limit cycle. The limit cycle is a

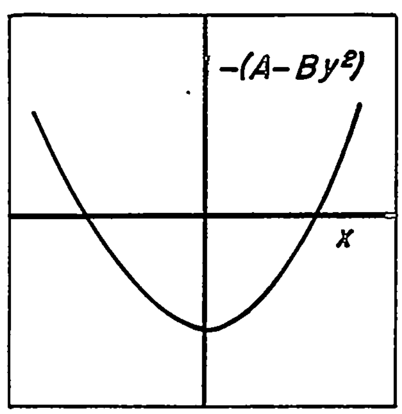
characteristic of self-sustained oscillations and is markedly different from the behavior of forced or undamped linear systems which can also exhibit closed trajectories in the phase plane. In these latter cases, however, the closed trajectories vary in size and shape for differing initial conditions.

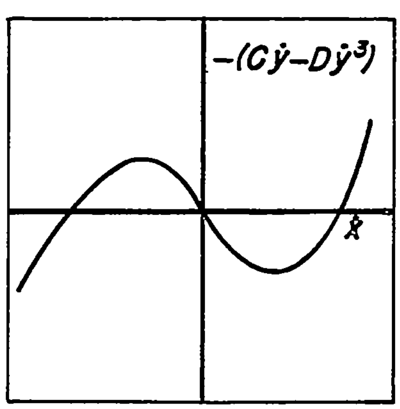
It is known that a large class of second-order systems capable of self-sustained oscillations can be described by either a Van der Pol equation or a Rayleigh equation. In the former equation the damping term is of the form

$$f(y,\dot{y}) = -(A - By^2)\,\dot{y} \tag{7}$$

and in the latter the damping term is

$$f(y,\dot{y}) = -(C\dot{y} - D\dot{y}^3) \tag{8}$$




The nature of the damping coefficient of Van der Pol's equation is shown in the first of the accompanying sketches, and the nature of the damping term of Rayleigh's equation is shown in the second sketch. In general, the restoring force need not be linear for a limit cycle to exist (see ref. 11). For the present analysis, it is assumed that the observed self-sustained oscillation can be described by a Van der Pol or a Rayleigh equation, and that the restoring forces are linear.

When, in equation (1), the restoring force is linear and the damping force is of the form of equation (7) the result is Van der Pol's equation

$$\ddot{y} - (A - By^2)\,\dot{y} + \omega_n^2 y = 0 \tag{9}$$

If the transformations

$$\left.\begin{aligned} t_1 &= \omega_n t \\ x &= \sqrt{\frac{B}{A}}\,y \\ \mu &= \frac{A}{\omega_n} \end{aligned}\right\} \tag{10}$$

are made, then equation (9) becomes

$$\frac{d^2x}{dt_1^2} - \mu(1 - x^2)\frac{dx}{dt_1} + x = 0 \tag{11}$$

Equation (30) is the form in which Van der Pol's equation is usually written. Similarly, if Rayleigh's equation,

$$\ddot{y} - (C\dot{y} - D\dot{y}^3) + \omega_n^2 y = 0 \tag{12}$$

is transformed by means of

$$\left.\begin{aligned} t_1 &= \omega_n t \\ x &= \sqrt{\frac{3\omega_n^2 D}{C}}\, y \\ \mu &= \frac{C}{\omega_n} \end{aligned}\right\} \tag{13}$$

The result is

$$\frac{d^2x}{dt_1^2} - \mu\left[\frac{dx}{dt_1} - \frac{1}{3}\left(\frac{dx}{dt_1}\right)^3\right] + x = 0 \tag{14}$$

This is the canonical form of Rayleigh's equation. Equations (11) and (14) are related by simple transformations, which will be shown next.

In equation (11), let $x = \frac{dz}{dt_1}$, and the result is

$$\frac{d^3z}{dt_1^3} - \mu\left[1 - \left(\frac{dz}{dt_1}\right)^2\right]\frac{d^2z}{dt_1^2} + \frac{dz}{dt_1} = 0 \tag{15}$$

Integrate equation (15) and the result is

$$\frac{d^2z}{dt_1^2} - \mu\left[\frac{dz}{dt_1} - \frac{1}{3}\left(\frac{dz}{dt_1}\right)^3\right] + z = \text{const.} \tag{16}$$

If the change of variable $z - \text{const.} = x$ is made, equation (16) becomes identical to equation (14). Thus, the Van der Pol equation has been transformed into a Rayleigh equation. Conversely, the Rayleigh equation, equation (14), can be transformed into a Van der Pol equation by the change of variable $z = \frac{dx}{dt_1}$. The result of this change of variable is differentiated once, which puts it into the form of equation (11). These manipulations show that the solution of Van der Pol's equation is the derivative of the solution of Rayleigh's equation.

From theoretical considerations (see ref. 8, pp. 197-199), it can be shown that for small amounts of damping (i.e., small values of $\mu$ or $\frac{A}{\omega_n}$), the phase-plane limit cycle of Van der Pol's equation is almost a circle of radius

$$R = \sqrt{\frac{4A}{B}} \tag{17}$$

and the limit cycle of Rayleigh's equation with small values of $\mu$ or $\frac{C}{\omega_n}$ is nearly a circle of radius

$$R = \sqrt{\frac{4C}{3\omega_n^2 D}} \tag{18}$$

When Van der Pol's and Rayleigh's equations are written in the standard forms, equations (11) and (14), then the phase plots are almost circles with radii $R = 2$ for small values of $\mu$.

In reference 12 a series expansion valid for all values of $\mu$, is given for the solution of a Van der Pol equation in the form of equation (11). From this series it can be shown that the maximum value of displacement does not change appreciably for increasing values of $\mu$, but remains nearly 2. By means of the simple relationship between Van der Pol's and Rayleigh's equations, and the above mentioned series solution, it can be shown that the variable $\frac{dx}{dt_1}$ in equation (14), Rayleigh's equation, changes but slightly as $\mu$ increases. This property is utilized in the inverse process, which will be described next.

The essence of the inverse technique is to compare the phase-plane plot of the experimental data to previously prepared phase plots of the standard forms of Van der Pol's or Rayleigh's equation. Such plots were obtained by solving equations (11) and (14) on the analogue computer for several values of $\mu$. (See figs. 1 and 2.) It will be observed in figures 1 that the maximum displacement does remain sensibly constant as $\mu$ increases, and in figures 2 that the maximum value of the

derivatives of the displacement does not change appreciably as $\mu$ increases. Note that figures 1 are quite different in appearance from figures 2, particularly at large values of $\mu$.

Suppose that a dynamic system is assumed describable by a Van der Pol equation. The frequency of oscillation of the response can be observed from the experimental responses, and it is approximately equal to $\omega_n$. Thus a value of $\omega_n$ is obtained. Next the phase plot of $x$ versus $\frac{\dot{x}}{\omega_n}$ is prepared. The maximum value of $x$ is noted, and it is set equal to $R$ (eq.(17)). This fixes a value for the ratio $\frac{B}{A}$. The next step is to plot the transformed phase portrait, the variables of which are

$$\left.\begin{aligned} x &= \sqrt{\frac{B}{A}}\, y \\ \frac{dx}{dt_1} &= \sqrt{\frac{B}{A}}\, \frac{\dot{y}}{\omega_n} \end{aligned}\right\} \tag{19}$$

This new phase plot is compared to the previously prepared phase plots of equation (11), figures 1, from which an approximate value of $\mu$ is determined. Now, using the relation

$$\mu = \frac{A}{\omega_n} \tag{20}$$

a value of $A$ can be computed. With this value, and equation (17), $B$ can be calculated. Thus the coefficients of the Van der Pol type of equation which approximately fits the experimental data are found.

If the given data are assumed to be responses of a Rayleigh equation, the technique for finding the coefficients of the damping term is similar to that of the foregoing discussion. A phase plot, $y$ versus $\frac{\dot{y}}{\omega_n}$, is made and the maximum value of $\frac{\dot{y}}{\omega_n}$ is observed. This value is then related to the ratio $\frac{C}{D}$ through equation (18). The transformed phase portrait, the variables being

$$\left.\begin{aligned} x &= \sqrt{\frac{3D\omega_n^2}{C}}\, y \\ \frac{dx}{dt_1} &= \sqrt{\frac{3D\omega_n^2}{C}}\, \frac{\dot{y}}{\omega_n} \end{aligned}\right\} \tag{21}$$

is drawn next, and it may be compared to figures 2 to ascertain an approximate value of $\mu$. Now

$$\mu = \frac{C}{\omega_n} \tag{22}$$

so C is determined. Substituting C into equation (18) will give a value for D. In this manner C and D are computed for the system which is describable by a Rayleigh equation.

An example showing application of the foregoing technique is presented in Appendix B. The responses used in this demonstration were obtained from wind-tunnel tests of an aircraft model mounted on a balance that permitted free oscillation in yaw and roll. Some difficulty is encountered in fixing a precise value of $\mu$ in this problem, due to the fact that fluctuating conditions in the wind tunnel resulted in the responses not being exactly periodic. The manner in which a compromise value of $\mu$ is chosen is given in the above-mentioned appendix.

## DATA AND METHODS REQUIRED FOR DETERMINATION OF NONLINEAR PARAMETERS

The procedure for reducing nonlinear response data to values of the parameters of the physical system being studied begins with knowledge or assumptions concerning the order and form of the differential equations describing the system. Such information can usually be obtained by careful consideration of the mechanical or electrical components of the system. Once suitable assumptions regarding the differential equations have been made, the nature of the various parameters can be investigated by methods such as those discussed and referred to herein.

### First-Order Systems

When the mass of a one-degree-of-freedom mechanical system is negligible with respect to the damping- and restoring-force coefficients, the system can be approximately represented by a first-order differential equation. Such a situation may also occur in an electrical system having very small inductance. These are known as degenerate systems. The parameters of degenerate systems can be determined by the derivative method, using one or more dynamic responses.

### Second-Order Systems

The derivative method can be applied in the case of second-order systems with any or all of the parameters unknown. The data recommended for the calculation of several parameters by this method are responses to sinusoidal inputs of various magnitudes, or modulated sinusoidal input responses. It was found, in the calculations given in a previous section, that the difficulty of ill-conditioned algebraic equations was not greatly reduced when responses to modulated inputs were used, however. The most satisfactory kind of data appeared to be several responses to sinusoidal inputs having greatly differing amplitudes.

When the restoring-force and damping-force coefficients are the unknowns, and if either or both are nonlinear, then the curve-fitting technique wherein responses to several step inputs are used can be applied with much more assurance of success than can the derivative method. It is of interest to note that this curve-fitting technique is independent of parameters in the forcing function, which may or may not be known.

If any one parameter, such as the restoring-force coefficient, the damping coefficient, or a coefficient in the forcing function is unknown, then the derivative method may be applied. A single dynamic response, such as a sinusoidal, or pulse or step input response is all the data that are needed. In the special case of nonlinear damping, which is partly dissipative and partly regenerative, limit cycles can occur, and this sort of nonlinearity is best studied by means of phase-plane techniques. One or more cycles of the self-sustained vibration of the response are the required data.

### Systems of Higher Order Than Second

In all cases of this type the derivative method seems to be the only practical approach. The number of responses required, preferably to sinusoidal inputs of different amplitudes, will depend entirely on the number of unknown parameters.

## CONCLUDING REMARKS

Methods have been presented for reducing response data from certain nonlinear dynamic systems to coefficients or system parameters. The nonconstant coefficients were assumed to be functions either of the dependent variable or of its time derivative. The methods could be separated into three categories, namely, the equations-of-motion methods, the response curve-fitting methods, and the phase-plane methods.

Of the first category, the derivative method, which was applicable to all the cases investigated, was found to be most suitable when only one unknown nonconstant coefficient was to be determined. As the number of unknown coefficients, constant or nonconstant, increased, the accuracy of the derivative method decreased due to the ill-conditioned nature of the simultaneous equations to be solved. Furthermore, as the order of the differential equation representing the system increased, the accuracy of the derivative method tended to decrease due to the difficulty of measuring or computing higher-order derivatives of the dependent variable.

The response curve-fitting method employed was the most satisfactory for second-order systems having more than one unknown coefficient. If the damping force is known to be partly degenerative and partly regenerative, such that self-sustained oscillations are observed, the phase-plane method was found to be a suitable method of determining the form of the nonlinear damping in certain types of second-order systems.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., Apr. 14, 1953

## APPENDIX A

## CALCULATION OF THE DISTORTION FACTOR

The purpose of this appendix is to demonstrate numerical techniques for determining the distortion factor of the responses of a dynamic system to sinusoidal inputs. The dynamic system under consideration here could be characteristic of a missile, and only the longitudinal motion of the missile will be studied. The equations of motion of this system, which are used in the numerical examples of Appendix B also, are presented here.

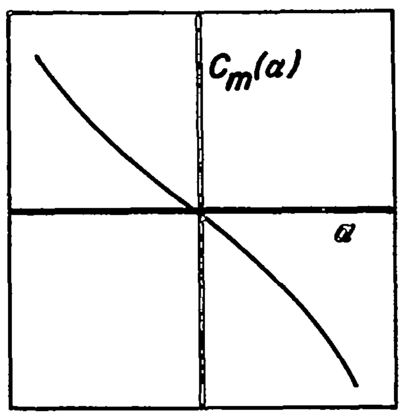
The longitudinal equations of motion of the missile are assumed in the form

$$\left.\begin{aligned} 2\tau\dot{\alpha} - 2\tau q - C_{Z_\alpha}\alpha &= -C_{Z_\delta}\delta \\ 2K\tau^2\dot{q} - C_m(\alpha) - C_{m_q}q &= C_{m_\delta}\delta \end{aligned}\right\} \tag{A1}$$

For a particular situation the coefficients in equation (A1) were calculated to be

$$\begin{aligned} \tau &= 2.188 \text{ sec} \\ K &= 0.001578 \\ C_{m_q} &= -0.0284 \text{ sec} \\ C_{Z_\delta} &= 0.030 \\ C_{m_\delta} &= 0.070 \\ C_{Z_\alpha} &= -5.32 \end{aligned}$$

The numerical values of the above coefficients are based upon $\alpha$ being measured in radians, $q$ in radians per second, and $\delta$ in degrees. The function $C_m(\alpha)$ is nonlinear and of the form shown in the sketch. Equations (A1) can now be written




$$\left.\begin{aligned} &\dot{\alpha} - q + 1.2157\alpha + 0.00685\delta = 0 \\ &\dot{q} + 1.8795q - 66.181C_m(\alpha) - 4.6327\delta = 0 \end{aligned}\right\} \tag{A2}$$

If $q$ is eliminated between the two equations (A2), the result is

$$\ddot{\alpha} + 3.0952\dot{\alpha} + 2.2850\alpha - 66.181C_m(\alpha) = 4.6198\delta - 0.00685\dot{\delta} \qquad \text{(A3)}$$

The data for the present computation were obtained by solving equation (A3) on the analogue computer with $\delta = \delta_o \sin \omega t$ where $\omega$ is $2\pi$ radians per second, and where values of $\delta_o$ used were $0.5^\circ$, $1^\circ$, $2^\circ$, $3^\circ$, and $3.5^\circ$. The $\dot{\delta}$ term was not included. The function $C_m(\alpha)$ as determined from wind-tunnel tests is shown in figure 3. The responses, to the sinusoidal inputs are presented in figure 4.

By use of a wave analyzer, the amplitudes of the first 10 harmonics of a steady-state cycle of each of the above responses were found. The amplitude of the first harmonic was adjusted to the value 100 so that the computation of the distortion factor would be simplified. The results are summarized in the chart that follows:

| Harmonic | Relative amplitudes of harmonics | | | | |
|---|---|---|---|---|---|
| | $\delta_o = 0.5^\circ$ | $\delta_o = 1^\circ$ | $\delta_o = 2^\circ$ | $\delta_o = 3^\circ$ | $\delta_o = 3.5^\circ$ |
| 1 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| 2 | 8.95 | 13.02 | 16.00 | 13.90 | 12.90 |
| 3 | 3.61 | 2.08 | 4.85 | 6.23 | 7.90 |
| 4 | 1.14 | 2.00 | 1.09 | 1.68 | 1.68 |
| 5 | .37 | .26 | .20 | 1.42 | 1.75 |
| 6 | .54 | .67 | .20 | .32 | .44 |
| 7 | .42 | .12 | .25 | .54 | .64 |
| 8 | .15 | .24 | .25 | .21 | .55 |
| 9 | .11 | .35 | .33 | .42 | .50 |
| 10 | .10 | .33 | .16 | .47 | .27 |
| Distortion factor | 9.8 | 13.3 | 16.7 | 15.4 | 15.3 |

As a check on the accuracy of the wave analyzer two exact sine waves were analyzed. The distortion factors for the two cases were 2.0 and 1.5.

By use of the same steady-state responses (fig. 4) the harmonic content of each was calculated by a digital integration process with the aid of the IBM computing equipment. The relative amplitudes of the first five harmonics of each response cycle are presented in the following chart:

| Harmonic | Relative amplitudes of harmonics | | | | |
|---|---|---|---|---|---|
| | $\delta_o = 0.5^\circ$ | $\delta_o = 1^\circ$ | $\delta_o = 2^\circ$ | $\delta_o = 3^\circ$ | $\delta_o = 3.5^\circ$ |
| 1 | 100.00 | 100.00 | 100.00 | 100.00 | 100.00 |
| 2 | 9.85 | 15.68 | 19.10 | 17.25 | 17.21 |
| 3 | .87 | 1.30 | 5.51 | 5.62 | 7.14 |
| 4 | 1.16 | 1.14 | .81 | 1.90 | .90 |
| 5 | .29 | .65 | .51 | .48 | 1.17 |
| Distortion factor | 10.0 | 15.8 | 19.8 | 18.3 | 18.6 |

An exact sine wave was analyzed in this manner and the percentage distortion was nearly zero.

The distortion factors were smaller when the wave analyzer was used than when the digital analysis was employed. There is probably no particular significance in this trend. It may be said, however, that there is much less chance for error in the digital process since the data to be used are read directly from the analogue computer records. For the wave analyzer it was necessary to cut a template in the shape of the wave to be analyzed, and there existed the probability of introducing inaccuracies at this step.

There are many methods for computing harmonic coefficients by hand. (See, for instance, refs. 13 and 14.) In the digital method used in the IBM computation, 24 points were taken over each of the steady-state response cycles. Then parabolas were fitted to overlapping triples of these points, thus giving 12 parabolas for each response. The integrals which define the Fourier coefficients were then integrated exactly, using these parabolic approximations to the function being analyzed, and the 12 integrals were summed to obtain each Fourier coefficient. This technique is quite accurate, but is probably too lengthy for hand computation. From the results of the previous analyses, it is apparent that the amplitude of the first harmonic divided by the amplitude of the fundamental closely approximates the value of the distortion factor. Thus any of the harmonic analyses which may be performed quickly by hand on desk calculating machines or slide rules probably give a good indication of the distortion factor, if enough points are used to assure a reasonably accurate determination of the first two harmonics beyond the fundamental.

# APPENDIX B

## ILLUSTRATIVE EXAMPLES

### Numerical Examples of the Derivative Method

Inasmuch as the success of the derivative method depends to a great extent upon the accuracy of the derivatives of the response, which are usually computed rather than measured, the numerical techniques employed herein for the data differentiation will be discussed briefly. The differentiations were performed by means of finite difference formulas, which are described in detail in reference 13. Since experimental data contain errors and irregularities, it is necessary to smooth the data mathematically before a numerical differentiation can be performed with accuracy. In the present work this smoothing has been carried out by means of a technique which is presented in reference 13, pages 303–316, and in references 15 and 16. Examples of the application of this method of smoothing, to be called herein the Whittaker method, will be found in these references.

In order to illustrate the effect of the Whittaker smoothing method, data which have been smoothed will be compared to the unsmoothed data in example 1. It will be shown that a primary effect of smoothing is that the third differences of the data are much smaller and more regular than the third differences of the unsmoothed data. The differentiation formulas therefore converge quite rapidly by using only the first three differences of the smoothed data.

Since the smoothing process is quite tedious, it has been performed on IBM computing equipment. The differentiations were all carried out by means of desk-type calculating machines, however.

Example 1, determination of nonlinear restoring-force parameter, all other parameters known, of a second-order system.– With the aid of the analogue computer, equation (A3) was solved using as the input $\delta = \sin 2\pi t$ and with $C_m(\alpha)$ given by the cubic

$$C_m(\alpha) = -1.2\alpha - 4.0\alpha^2 - 90\alpha^3 \tag{B1}$$

The response $\alpha$ is shown in figure 5. This response is now regarded as the given data, and $C_m(\alpha)$ is to be calculated. All of the other parameters of equation (A3) will be assumed known.

If several cycles of the response were available in the steady-state range, then an average of corresponding ordinates for several such cycles would undoubtedly reduce some of the irregularities in the

data. In the present example, however, there were not sufficient data for this, but the first five points were averaged with the first five points of the following cycle so that there would be more assurance that the numerically computed derivatives of the assumed cycle of data would appear as derivatives of a truly periodic response.

Ordinates of $\alpha$ were read from figure 5 at intervals of 0.02 second, starting at $t = 1.84$ seconds and ending at $t = 2.92$ seconds. These data are recorded in column 1 of table I. The first and last five values were then paired and averaged in the manner mentioned in the preceding paragraphs. These averaged end values are given in column 2 of table I. With these new end ordinates the data were assumed to be a representative steady-state cycle of $\alpha$. The first three differences of the response $\alpha$ were taken, and in column 3 of table I the third differences were recorded, and for purposes of comparison later on, the derivative $d\alpha/dt$ was taken and recorded in column 4.

Whittaker's smoothing method was then applied to the $\alpha$ data, using 0.05 as the value of the parameter $\epsilon$ in the smoothing formula. The smoothed values are denoted by the symbol $\alpha_s$. In table I and columns 5, 6, and 7, these values are tabulated, along with their third differences and the averaged end values of $\alpha_s$. Note that the third differences of the smoothed data, designated by the symbol $\Delta^3\alpha_s$, are much smaller than those of the unsmoothed data, while the actual deviation of smoothed from unsmoothed values is very slight.

The $\alpha_s$ data were next differentiated and the result was denoted $d\alpha_s/dt$. These data are tabulated in table I, column 8. After smoothing $d\alpha_s/dt$, the result being designated by the symbol $(d\alpha_s/dt)_s$, and after pairing and averaging the first and last five values of the smoothed data, a second derivative $d(d\alpha_s/dt)_s/dt$ was taken. For comparative purposes, the derivative of the unsmoothed data $d\alpha_s/dt$ was also computed. In columns 9, 10, 11, and 12 of table I the data pertaining to the second derivative of $\alpha$ are tabulated.

In figure 6 the derivatives of $\alpha$ and $\alpha_s$ are plotted, and in figure 7 the derivatives of $d\alpha_s/dt$ and $(d\alpha_s/dt)_s$ are likewise shown. These comparisons serve to show the need for, and effect of, the smoothing of a set of data prior to performing a numerical differentiation.

The derivatives that have been computed will now be used in connection with equation (A3) to demonstrate the derivative method of data reduction.

The forcing function of equation (A3) is

$$H(t) = 4.6198\ \delta - 0.00685\ \dot{\delta} \tag{B2}$$

which is, in this example,

$$H(t) = 4.6198 \sin 2\pi t - 0.0431 \cos 2\pi t \tag{B3}$$

Solving equation (A3) for $C_m(\alpha)$ gives

$$C_m(\alpha) = \frac{1}{66.181} \left[\ddot{\alpha} + 3.0952\,\dot{\alpha} + 2.2850\,\alpha - H(t)\right] \tag{B4}$$

In equation (B4) the data of columns 5, 10, and 12 of table I were used for $\alpha$, $\dot{\alpha}$, and $\ddot{\alpha}$, respectively. The forcing function, given by equation (B2), was tabulated in column 13 of table I. By use of these data $C_m(\alpha)$ was first computed as a function of time. Since each value so found corresponds to a value of $\alpha$, a plot of $C_m(t)$ versus $\alpha(t)$ gave the desired function, $C_m(\alpha)$. In column 14 of table I $C_m(\alpha)$ is given.

The function $C_m(\alpha)$ is plotted against $\alpha$ in figure 8. If the results were regarded as mathematically exact, then it might be concluded that $C_m(\alpha)$ is double valued. It is known, however, that $C_m(\alpha)$ is single valued, so this apparent hysteresis is, in fact, due to systematic distortion injected into the calculation in the smoothing and differentiating processes. For a more direct comparison of this result with the known $C_m(\alpha)$ (eq. (B1)), a cubic was fitted by least-squares methods to the computed function. This cubic is

$$C_m(\alpha) = -1.234\,\alpha - 4.004\,\alpha^2 - 40.456\,\alpha^3 \tag{B5}$$

The large discrepancy between the coefficients of $\alpha^3$ in equations (B5) and (B1) is undoubtedly caused by the strong effects of $\alpha^3$ which are only felt at large values of $\alpha$, and the data of this example do not extend sufficiently far from zero to afford a more accurate value for this coefficient. Note in figure 8 that the function $C_m(\alpha)$ of equation (B5) is quite accurate except at large values of $\alpha$.

As a further example in computing a single unknown parameter by the derivative method, the techniques described in the first example will be applied to transient responses of equations (A2). The data were obtained by solving equations (A2) on an analogue computer, using for $C_m(\alpha)$ the function given in figure 3. The oscillation was initiated by an initial displacement in $\alpha$ of $8^\circ$ (0.1396 radians). The responses were thus free oscillations, and the forcing term $\delta$ was not present in the computations to determine $C_m(\alpha)$.

In figure 9 are shown portions of the responses $q$ and $\alpha$ of the above system. The unknown function $C_m(\alpha)$ can be calculated from the second of equations (A2), so only one derivative of $q$ was required, and none of $\alpha$. The responses in $q$ and $\alpha$ were read at intervals of 0.02 second for a 1-second range starting at $t = 0$, and these results are given in table II. The $q$ data were smoothed by Whittaker's method, with $\epsilon = 0.1$, and the first derivative was taken. These results are also presented in table II. In figure 10 the derivative $dq_s/dt$ is plotted, and it is apparent that the value of $\epsilon$ of 0.1 is not too large. The accuracy with which the unsmoothed $\alpha$ and $q$ can be read from the analogue computer traces in this case justifies the use of a value larger than was used in the first example.

The function $C_m(\alpha)$ was found from the equation

$$C_m(\alpha) = \frac{1}{66.181} (\dot{q} + 1.8795\ q) \tag{B6}$$

The results of this calculation are given in table II and are plotted in figure 11.

Example 2, determination of amplifier characteristic curve in autopilot servo system, all other parameters known.- In reference 17 the dynamic behavior of an autopilot servo system was simulated on the analogue computer. An amplifier in the servo system was known to possess a saturation type of nonlinear characteristic. In order to account for certain time lags in the dynamic system it was found necessary to introduce an exponential lag operator (see ref. 10) into the equations of motion. A three-term approximation was made to the lag operator, which had the effect of raising the differential equation from second to fourth order. By use of this fourth-order differential equation and experimental responses obtained in bench tests of the autopilot, the problem in this example was to calculate the nonlinear amplifier characteristic when all the other parameters of the system are known.

After the approximation to the lag operator has been made, the open-loop equation is

$$\frac{\tau_D^2\tau_m}{2P_fk_fk_m}\frac{d^4v_f}{dt^4} + \frac{\tau_D\tau_m + \frac{\tau_D^2}{2}}{P_fk_fk_m}\frac{d^3v_f}{dt^3} + \frac{\tau_D+\tau_m}{P_fk_fk_m}\frac{d^2v_f}{dt^2} + \frac{1}{P_fk_fk_m}\frac{dv_f}{dt} = I(v_e) \tag{B7}$$

where $v_f$ is the output voltage and $v_e$ the error voltage. The error and output are related to the input by the relation $v_e = v_i - v_f$, where $v_i$ is the input voltage. The parameters $\tau_D$, $\tau_m$, $P_f$, $k_f$, and $k_m$ are all constants, and

$$I(v_e) = P_a A_a v_e \tag{B8}$$

where $P_a A_a$ is the nonconstant slope of the amplifier characteristic. In a certain case the parameters in equation (14) were

$$\left.\begin{aligned} P_f &= 0.24 \\ k_f &= 12.8 \\ k_m &= 0.063 \\ \tau_m &= 0.052 \\ \tau_D &= 0.009 \end{aligned}\right\} \tag{B9}$$

When these values are used, equation (B7) becomes

$$0.000011 \frac{d^4 v_f}{dt^4} + 0.00263 \frac{d^3 v_f}{dt^3} + 0.315 \frac{d^2 v_f}{dt^2} + 5.17 \frac{dv_f}{dt} - I(v_e) = 0 \tag{B10}$$

The response of the servo system to an input voltage $v_i = 1.56 \sin 2\pi t$ was obtained. In figure 12, $v_i$, $v_f$, and $v_e$ are plotted. The function $v_f$, does not appear at a glance to be the response of a nonlinear system. By use of the digital harmonic analysis technique described in Appendix A, the distortion factor was computed and found to be 7.68.

After smoothing $v_f$ by Whittaker's method with $\epsilon = 0.25$, the first derivative of $v_f$ was found, using the difference formulas previously cited. Three more derivatives were calculated, smoothing each one before the next derivative was taken. These four derivatives are shown in figure 13. It is apparent that even with the large magnitudes attained by the fourth derivative, its contribution in equation (17) was very small. The four derivatives of $v_f$ were substituted into equation (17). Then $I(v_e)$ was computed in the same manner as was $C_m(\alpha)$ in the previous two examples. The results of this calculation and all the other data for this example are given in table III. In figure 14, $I(v_e)$ is plotted as calculated here and also as found by static tests of the autopilot. The false hysteresis effect is present, but the actual saturation trend of the characteristic is preserved in the function as computed by the derivative method.

Example 3, determination of two unknown parameters by the derivative method.- The extension of the derivative method to the analysis of nonlinear response data for two unknown coefficients will now be demonstrated with numerical examples. The data were obtained by solving equation (A3) on the analogue computer, where $C_m(\alpha)$ is again given by figure 3, and the $\delta$ term is neglected. For the purpose of the following computations, equation (A3) was put into the form

$$\ddot{\alpha} + b\dot{\alpha} + k\alpha = C_o\delta \tag{B11}$$

where $b$, $k$, and $C_o$ are constants. Equation (B11) is therefore linear, but it was used to describe the behavior of the system under consideration only over small ranges of values of $\alpha$. The coefficient $b$ was assumed known from other considerations, and $k$ and $C_o$ were calculated as constants for several values of $\alpha$, using the technique previously described. The computed values of $k$ and $C_o$ could then be related to the values of $\alpha$ at which they are determined in order to ascertain their variation as $\alpha$ is varied. In this particular example the function $C_m(\alpha)$ was calculated in addition to $k$, using the relation

$$C_m(\alpha) = \frac{2.285 - k}{66.181}\alpha \tag{B12}$$

Equation (B12) was obtained from the restoring-moment terms of equation (A3), and $k$ is the number computed above for the value of $\alpha$ at which $C_m(\alpha)$ was to be determined.

A set of data was obtained for the phase-modulated sinusoidal input

$$\delta = 1.75 \sin (2\pi t + \pi \sin 0.3\pi t) \tag{B13}$$

As was mentioned in a previous section, this sort of response data has the advantage that one run of the dynamic system is sufficient. The data so obtained were smoothed and differentiated. Then the response, its first two derivatives, and the forcing function were plotted to the same time scale in the manner described previously. At values of $\alpha$ of 0.02, 0.06, and 0.08 radians, lines were drawn through the response curve parallel to the time axis. At each instant where a line cuts the $\alpha$ curve, the corresponding values of the two derivatives and the forcing function were read. For each value of $\alpha$, then, the quantities so read were substituted into equation (B11). The resulting sets of equations were solved by least squares for $k$ and $C_o$. These calculations are summarized in the following table, where $k$ is also related to $C_m(\alpha)$ by equation (B12):

| $\alpha$ | $C_o$ | $k$ | $C_m(\alpha)$ from equation (B12) | $C_m(\alpha)$ from figure 3 |
|---|---|---|---|---|
| 0.02 | 4.861 | 111.42 | −0.033 | −0.030 |
| .06 | 6.082 | 161.51 | −.144 | −.120 |
| .08 | 5.446 | 150.31 | −.179 | −.180 |

The correspondence between the calculated values of $C_m(\alpha)$ and the values taken from figure 1 is indicated in the above chart, and the true value of $C_o$ is 4.62.

As a second example, an amplitude-modulated sine wave was used for $\delta$; that is,

$$\delta = 2 \sin 2\pi t + \sin 2.2\pi t \tag{B14}$$

Again $b$ was assumed known, and calculations similar to these above were made to determine values of $k$ and $C_o$ for several values of $\alpha$. These results are given in the following table:

| $\alpha$ | $C_o$ | $k$ | $C_m(\alpha)$ from equation (B12) | $C_m(\alpha)$ from figure 3 |
|---|---|---|---|---|
| 0.025 | 2.669 | 49.25 | -0.018 | -0.032 |
| .050 | 3.806 | 101.22 | -.075 | -.095 |
| -.075 | 5.154 | 105.94 | .118 | .110 |

Responses of this same system to sinusoidal inputs of various amplitudes were obtained, and a calculation similar to the ones discussed above was made. The values of $C_o$ were more accurate in the case of the responses to sinusoidal inputs, but $C_m(\alpha)$ was of about the same order of accuracy as before.

The results of the above calculations are not entirely satisfactory. The difficulty of ill-conditioning of the least-squares normal equations was present, although modulated inputs were tried.

From these examples it is apparent that the accuracy of the derivative method of nonlinear data reduction may be poor for the case of two unknown parameters, due to the seemingly unavoidable ill-conditioned equations that must be solved. For systems of order higher than second, however, this method may yield results which can then be further refined.

## Numerical Example of the Curve-Fitting Technique

In order to demonstrate the simple period and damping technique of analysis of nonlinear response data, the following example is presented. The data were obtained by solving equation (A3), with the $\dot{\delta}$ term omitted, on the analogue computer, using step inputs of magnitudes $-0.5^\circ$, $-1^\circ$, $-2^\circ$, $-3^\circ$, $-4^\circ$, $0.25^\circ$, $0.5^\circ$, $0.75^\circ$, $1^\circ$, $2^\circ$, $3^\circ$, and $4^\circ$. The function $C_m(\alpha)$ that was used is that given in figure 3. With the assumption that each response is linear, $t_{1/2}$ and $P$ were computed for the responses. It was found that $t_{1/2}$ is nearly constant, and the values of $2\zeta\omega_n$, the damping coefficient, are tabulated here. The average value for $2\zeta\omega_n$ is 2.91, which differs by about 5 percent from the correct constant value, 3.095. This indicates that the technique gives reasonably accurate results.

| $\delta_o$ | $2\zeta\omega_n$ |
|---|---|
| 4.0 | 3.11 |
| 3.0 | 3.00 |
| 2.0 | 2.91 |
| 1.0 | 2.89 |
| .75 | 2.85 |
| .5 | 2.85 |
| .25 | 2.78 |
| -.5 | 3.15 |
| -1.0 | 3.00 |
| -2.0 | 2.89 |
| -3.0 | 2.96 |
| -4.0 | 2.89 |

It was observed that the period $P$ of the oscillations varies as the amplitude of the input varies. Thus $\omega_n^2$ varies. In order to make a comparison with $C_m(\alpha)$ of the variation of $\omega_n^2$ let

$$\omega_n^2\alpha = -66.181 \frac{\partial C_m^*}{\partial \alpha} \alpha \qquad \text{(B15)}$$

The numbers $\frac{\partial C_m^*}{\partial \alpha}$ are calculated and then plotted against the corresponding trim values of $\alpha$. This curve, given in figure 15, was then integrated in an approximate manner and the result was designated by the symbol $C_m^*(\alpha)$. This function is related to $C_m(\alpha)$ by the equation

$$-66.181\ C_m^*(\alpha) = 2.285\ \alpha - 66.181\ C_m(\alpha) \qquad \text{(B16)}$$

The function $C_m(\alpha)$, as calculated from equation (B16), is plotted in figure 16. The $C_m(\alpha)$ of figure 3 is also given in figure 16 for comparison. Considering the basic simplicity of this calculation, the results are very good.

## Numerical Example of the Phase-Plane Method

As an example of the application of the phase-plane technique for determining the nonlinear damping involved in a self-sustained oscillation, a numerical problem will be solved. In wind-tunnel tests performed upon an aircraft model, the model was observed to initiate a self-sustained lateral oscillation about its mounting as the speed of the wind in the tunnel reached a certain level. Several cycles of this oscillation were recorded and read at intervals of 0.005 second, and these data are tabulated in table IV. The above data were smoothed by Whittaker's method with $\epsilon = 0.25$, and then differentiated. The results of this calculation are also presented in table IV.

The differential equation of the dynamical system was assumed to be of the form

$$I_Z \ddot{\psi} + f(\psi,\psi) + (N_\psi + N_\beta)\psi = 0 \tag{B17}$$

The constants $I_Z$ and $(N_\psi + N_\beta)$ were known to be

$$I_Z = 0.944 \text{ in-lb-sec}^2\text{/radian}$$

$$N_\psi + N_\beta = 742.5 \text{ in-lb/radian}$$

and $f(\psi,\psi)$ was to be calculated. If the notation of the previous discussion is adhered to, then

$$\omega_n = \sqrt{\frac{N_\psi + N_\beta}{I_Z}} = 28.05 \text{ radians/sec}$$

The phase portrait, $\psi$ versus $\dot{\psi}/\omega_n$, was drawn next. (See fig. 17.) Note that the phase curve is not a closed limit cycle. This is due to variations in the data, caused undoubtedly by fluctuating conditions in the wind tunnel. From the general trend of this phase portrait, namely the predominantly negative slopes in the first and third quadrants, and also from physical considerations that the damping should be a function only of $\psi$, it was concluded that a Rayleigh equation can be used to describe the motion. The damping term of equation (B17) was therefore assumed to have the form

$$\frac{f(\dot{\psi},\psi)}{I_Z} = -(C\dot{\psi} - D\dot{\psi}^3) \tag{B18}$$

From figure 17 the average maximum value of $\dot{\psi}/\omega_n$ was found to be approximately

$$R = 0.391(57.3) = 22.40$$

The conversion factor 57.3 is required, since $\psi$ is given in degrees and $\omega_n$ has the dimensions of radians per second, and $R$ must be dimensionless. Then, using equation (18), it was found that

$$\frac{D}{C} = 36.40 \tag{B19}$$

By means of equations (21), the new transformed phase portrait was constructed. The variables were

$$\left.\begin{aligned} x &= 5.10\psi \\ \frac{dx}{dt_1} &= 5.10\,\frac{\dot{\psi}}{\omega_n} \end{aligned}\right\} \tag{B20}$$

This phase plot (fig. 18) was compared to figure 2, and it was found that the phase curve of $\mu = 0.6$ fits the experimental phase portrait. This curve is superimposed upon figure 18 for comparison. From the value of $\mu$ of 0.6 and the relation $\mu = C/\omega_n$ the value of $C$ was found to be 16.82, and then from equation (B19), D was calculated. It was 612. With these two numbers, equation (B17) can be written

$$0.944\ddot{\psi} - (15.9\dot{\psi} - 578\dot{\psi}^3) + 742.5\psi = 0 \tag{B21}$$

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TABLE I.- CALCULATIONS FOR EXAMPLE 1

[Determination of $C_m(\alpha)$ from response to sinusoidal input]

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| t | $10^4\alpha$ | Averaged end values $10^4\alpha$ | $10^4\Delta^4\alpha$ | $10^4 \frac{d\alpha}{dt}$ | $10^4\alpha_s$ | Averaged end values $10^4\alpha_s$ | $10^2\Delta^4\alpha_s$ | $10^4 \frac{d\alpha_s}{dt}$ | $10^3 \frac{d}{dt}\left(\frac{d\alpha_s}{dt}\right)$ | $10^4 \left(\frac{d\alpha_s}{dt}\right)_s$ | Averaged end values of column 10 | $10^3 \frac{d}{dt}\left(\frac{d\alpha_s}{dt}\right)_s$ | H(t) | $C_m(\alpha)$ |
| 1.84 | -720 | -718 | -1 | -1129 | -717.6 | -717.4 | 1.9 | -1040 | 2340 | -1030 | -1030 | 2310 | -4.407 | 0.0942 |
| 1.86 | -735 | -733 | 7 | -492 | -732.6 | -732.8 | -1.5 | -504 | 2731 | -519 | -524 | 2607 | -4.198 | .0978 |
| 1.88 | -741 | -738 | -8 | -50 | -737.9 | -737.5 | -.5 | 27 | 2683 | 18 | 1 | 2673 | -3.924 | .0971 |
| 1.90 | -738 | -734 | 0 | 604 | -731.6 | -731.8 | 4.0 | 543 | 2471 | 562 | 544 | 2753 | -3.587 | .0958 |
| 1.92 | -718 | -714 | 6 | 1229 | -715.0 | -715.8 | -5.0 | 1040 | 2737 | 1112 | 1099 | 2788 | -3.194 | .0931 |
| 1.94 | -686 | --- | -4 | 1575 | -686.6 | --- | 1.6 | 1669 | 3110 | 1652 | --- | 2733 | -2.750 | .0882 |
| 1.96 | -650 | --- | -1 | 2142 | -648.4 | --- | -1.7 | 2244 | 2573 | 2189 | --- | 2628 | -2.263 | .0819 |
| 1.98 | -600 | --- | -10 | 2771 | -599.4 | --- | .5 | 2710 | 2538 | 2704 | --- | 2513 | -1.741 | .0749 |
| 2.00 | -540 | --- | 14 | 3321 | -540.0 | --- | -2.9 | 3193 | 2387 | 3191 | --- | 2344 | -1.191 | .0665 |
| 2.02 | -469 | --- | -12 | 3758 | -471.9 | --- | 1.9 | 3639 | 2080 | 3640 | --- | 2156 | -.622 | .0574 |
| 2.04 | -397 | --- | 2 | 3967 | -394.6 | --- | -3.3 | 4027 | 1832 | 4051 | --- | 1983 | -.043 | .0472 |
| 2.06 | -310 | --- | 5 | 4467 | -311.0 | --- | -1.0 | 4391 | 1814 | 4409 | --- | 1710 | .536 | .0373 |
| 2.08 | -222 | --- | -5 | 4596 | -219.2 | --- | -1.8 | 4730 | 1398 | 4727 | --- | 1398 | 1.107 | .0248 |
| 2.10 | -125 | --- | 10 | 5000 | -122.5 | --- | -2.2 | 4944 | 608 | 4943 | --- | 968 | 1.585 | .0134 |
| 2.12 | -20 | --- | -11 | 5104 | -21.9 | --- | -2.2 | 5095 | 893 | 5111 | --- | 558 | 2.188 | -.0009 |
| 2.14 | 80 | --- | 3 | 5129 | 80.8 | --- | -1.8 | 5152 | 31 | 5157 | --- | 10 | 2.681 | -.0159 |
| 2.16 | 185 | --- | -10 | 5142 | 183.4 | --- | -2.4 | 5090 | -563 | 5115 | --- | -476 | 3.131 | -.0299 |
| 2.18 | 284 | --- | 12 | 4904 | 283.7 | --- | -3.0 | 4930 | -1033 | 4960 | --- | -1088 | 3.532 | -.0456 |
| 2.20 | 380 | --- | -7 | 4467 | 379.9 | --- | -2.4 | 4670 | -1655 | 4678 | --- | -1733 | 3.878 | -.0616 |
| 2.22 | 463 | --- | -10 | 4103 | 469.6 | --- | -2.4 | 4270 | -2343 | 4264 | --- | -2428 | 4.162 | -.0780 |
| 2.24 | 545 | --- | -2 | 3971 | 549.6 | --- | -3.7 | 3733 | -2956 | 3708 | --- | -3126 | 4.380 | -.0942 |
| 2.26 | 619 | --- | 2 | 3300 | 618.1 | --- | -.6 | 3083 | -3663 | 3017 | --- | -3777 | 4.530 | -.1093 |
| 2.28 | 675 | --- | -3 | 2300 | 672.1 | --- | -1.3 | 2268 | -4410 | 2205 | --- | -4313 | 4.608 | -.1222 |
| 2.30 | 711 | --- | 20 | 1354 | 708.1 | --- | 1.3 | 1340 | -4783 | 1304 | --- | -4667 | 4.614 | -.1317 |
| 2.32 | 729 | --- | -8 | 304 | 725.5 | --- | .5 | 373 | -4878 | 354 | --- | -4782 | 4.564 | -.1371 |
| 2.34 | 726 | --- | -15 | -650 | 723.0 | --- | 4.8 | -598 | -4777 | -591 | --- | -4624 | 4.407 | -.1390 |
| 2.36 | 705 | --- | 12 | -1179 | 701.9 | --- | 1.6 | -1530 | -4355 | -1479 | --- | -4213 | 4.198 | -.1316 |
| 2.38 | 665 | --- | 5 | -2088 | 662.7 | --- | 4.4 | -2319 | -3576 | -2262 | --- | -3582 | 3.924 | -.1217 |
| 2.40 | 611 | --- | 2 | -3071 | 610.2 | --- | 2.0 | -2943 | -2747 | -2902 | --- | -2803 | 3.587 | -.1080 |
| 2.42 | 545 | --- | 8 | -3504 | 546.0 | --- | 5.1 | -3419 | -2001 | -3379 | --- | -1984 | 3.194 | -.0919 |
| 2.44 | 472 | --- | -6 | -3817 | 474.5 | --- | .4 | -3737 | -1115 | -3683 | --- | -1137 | 2.750 | -.0743 |
| 2.46 | 394 | --- | 11 | -3833 | 397.7 | --- | 2.8 | -3868 | -350 | -3841 | --- | -431 | 2.263 | -.0573 |
| 2.48 | 319 | --- | -12 | -3847 | 320.7 | --- | -.4 | -3858 | 263 | -3869 | --- | 127 | 1.741 | -.0414 |
| 2.50 | 241 | --- | 11 | -3696 | 243.9 | --- | -.1 | -3775 | 627 | -3802 | --- | 503 | 1.191 | -.0273 |
| 2.52 | 171 | --- | -9 | -3596 | 170.1 | --- | -1.0 | -3623 | 750 | -3680 | --- | 688 | .622 | -.0156 |
| 2.54 | 97 | --- | -2 | -3550 | 98.9 | --- | -2.0 | -3493 | 526 | -3537 | --- | 720 | .043 | -.0060 |
| 2.56 | 30 | --- | 3 | -3271 | 30.2 | --- | -2.8 | -3402 | 554 | -3399 | --- | 650 | -.536 | .0021 |
| 2.58 | -35 | --- | -4 | -3254 | -37.0 | --- | -.2 | -3269 | 675 | -3261 | --- | 520 | -1.107 | .0091 |
| 2.60 | -100 | --- | -1 | -3171 | -100.7 | --- | .4 | -3195 | 313 | -3192 | --- | 373 | -1.585 | .0143 |
| 2.62 | -162 | --- | 12 | -3103 | -163.7 | --- | 0 | -3138 | 82 | -3130 | --- | 255 | -2.188 | .0217 |
| 2.64 | -225 | --- | -20 | -3246 | -226.2 | --- | -.1 | -3104 | 215 | -3086 | --- | 200 | -2.681 | .0283 |
| 2.66 | -290 | --- | 0 | -2967 | -287.8 | --- | 2.3 | -3057 | 180 | -3046 | --- | 203 | -3.131 | .0345 |
| 2.68 | -345 | --- | 10 | -2942 | -348.5 | --- | -1.7 | -3024 | 351 | -3000 | --- | 279 | -3.532 | .0424 |
| 2.70 | -410 | --- | 3 | -3042 | -408.4 | --- | 2.6 | -2920 | 561 | -2929 | --- | 440 | -3.878 | .0501 |
| 2.72 | -465 | --- | 6 | -2808 | -465.2 | --- | .5 | -2809 | 528 | -2820 | --- | 657 | -4.162 | .0580 |
| 2.74 | -520 | --- | -16 | -2715 | -520.6 | --- | 1.5 | -2683 | 841 | -2663 | --- | 923 | -4.380 | .0659 |
| 2.76 | -572 | --- | 32 | -2333 | -572.0 | --- | 2.8 | -2466 | 1133 | -2449 | --- | 1218 | -4.530 | .0734 |
| 2.78 | -615 | --- | -28 | -2368 | -618.9 | --- | -2.0 | -2213 | 1579 | -2175 | --- | 1527 | -4.608 | .0804 |
| 2.80 | -665 | --- | 16 | -1892 | -659.8 | --- | 2.8 | -1824 | 1993 | -1839 | --- | 1835 | -4.614 | .0866 |
| 2.82 | -690 | --- | -3 | -1275 | -691.9 | --- | 1.9 | -1438 | 1925 | -1447 | --- | 2009 | -4.564 | .0902 |
| 2.84 | -715 | -718 | 1 | -1129 | -717.2 | -717.4 | -2.2 | -1040 | 2340 | -1009 | -1030 | 2310 | -4.407 | .0942 |
| 2.86 | -730 | -733 | 7 | -492 | -734.9 | -732.8 | 1.0 | -504 | 2731 | -529 | -524 | 2607 | -4.198 | .0978 |
| 2.88 | -735 | -738 | --- | -58 | -737.1 | -737.5 | --- | 27 | 2683 | -16 | 1 | 2673 | -3.924 | .0971 |
| 2.90 | -729 | -734 | --- | 604 | -732.0 | -731.8 | --- | 543 | 2471 | 526 | 544 | 2753 | -3.587 | .0958 |
| 2.92 | -710 | -714 | --- | 1575 | -716.6 | -715.8 | --- | 1040 | 2835 | 1086 | 1099 | 2788 | -3.194 | .0931 |

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TABLE II.– CALCULATIONS FOR EXAMPLE 1

[Determination of $C_m(\alpha)$ from transient response data]

| t | 100 q | 100 $q_s$ | 100 $\frac{dq_s}{dt}$ | $10^3\alpha$ | $C_m(\alpha)$ | t | 100 q | 100 $q_s$ | 100 $\frac{dq_s}{dt}$ | $10^3\alpha$ | $C_m(\alpha)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | –3.4 | –2819 | 140 | –0.4269 | 0.52 | 32 | 31.7 | –854 | 61 | –0.1200 |
| .02 | –57 | –54.0 | –2255 | 127 | –.3421 | .54 | 15 | 14.3 | –877 | 64 | –.1285 |
| .04 | –94 | –93.2 | –1675 | 112 | –.2796 | .56 | –4 | –3.0 | –845 | 63 | –.1285 |
| .06 | –120 | –121.2 | –1135 | 92 | –.2059 | .58 | –20 | –19.1 | –746 | 59 | –.1181 |
| .08 | –141 | –138.9 | –639 | 61 | –.1360 | .60 | –34 | –32.8 | –616 | 52 | –.1024 |
| .10 | –147 | –147.1 | –194 | 32 | –.0710 | .62 | –44 | –43.6 | –462 | 44 | –.0822 |
| .12 | –146 | –147.1 | 180 | 3 | –.0146 | .64 | –52 | –51.2 | –296 | 33 | –.0591 |
| .14 | –138 | –140.3 | 490 | –25 | .0341 | .66 | –55 | –55.5 | –145 | 23 | –.0377 |
| .16 | –126 | –127.8 | 749 | –51 | .0767 | .68 | –56 | –57.0 | –13 | 11 | –.0181 |
| .18 | –112 | –110.6 | 970 | –72 | .1152 | .70 | –56 | –56.1 | 100 | 0 | –.0008 |
| .20 | –90 | –89.3 | 1144 | –90 | .1475 | .72 | –53 | –53.1 | 196 | –11 | .0145 |
| .22 | –66 | –65.0 | 1274 | –103 | .1740 | .74 | –49 | –48.4 | 275 | –21 | .0278 |
| .24 | –40 | –38.6 | 1341 | –113 | .1917 | .76 | –42 | –42.2 | 338 | –29 | .0391 |
| .26 | –14 | –11.7 | 1333 | –115 | .1981 | .78 | –35 | –35.0 | 380 | –38 | .0475 |
| .28 | 17 | 14.2 | 1238 | –112 | .1911 | .80 | –27 | –27.1 | 410 | –43 | .0543 |
| .30 | 40 | 37.6 | 1078 | –105 | .1736 | .82 | –19 | –18.7 | 428 | –46 | .0594 |
| .32 | 58 | 57.0 | 866 | –92 | .1470 | .84 | –10 | –10.1 | 428 | –47 | .0618 |
| .34 | 78 | 72.0 | 633 | –79 | .1161 | .86 | –2 | –1.7 | 409 | –48 | .0613 |
| .36 | 83 | 82.3 | 300 | –59 | .0838 | .88 | 6 | 6.2 | 383 | –47 | .0596 |
| .38 | 87 | 88.1 | 187 | –42 | .0533 | .90 | 14 | 13.5 | 339 | –44 | .0553 |
| .40 | 89 | 89.9 | –5 | –21 | .0248 | .92 | 20 | 19.7 | 286 | –38 | .0488 |
| .42 | 87 | 88.0 | –182 | –2 | –.0025 | .94 | 25 | 24.9 | 229 | –35 | .0417 |
| .44 | 82 | 82.7 | –349 | 16 | –.0293 | .96 | 29 | 28.8 | 162 | –28 | .0327 |
| .46 | 75 | 74.1 | –507 | 31 | –.0556 | .98 | 31 | 31.4 | 97 | –21 | .0249 |
| .48 | 63 | 62.5 | –654 | 44 | –.0811 | 1.00 | 33 | 32.8 | 42 | –13 | .0157 |
| .50 | 50 | 48.1 | –778 | 54 | –.1039 | | | | | | |

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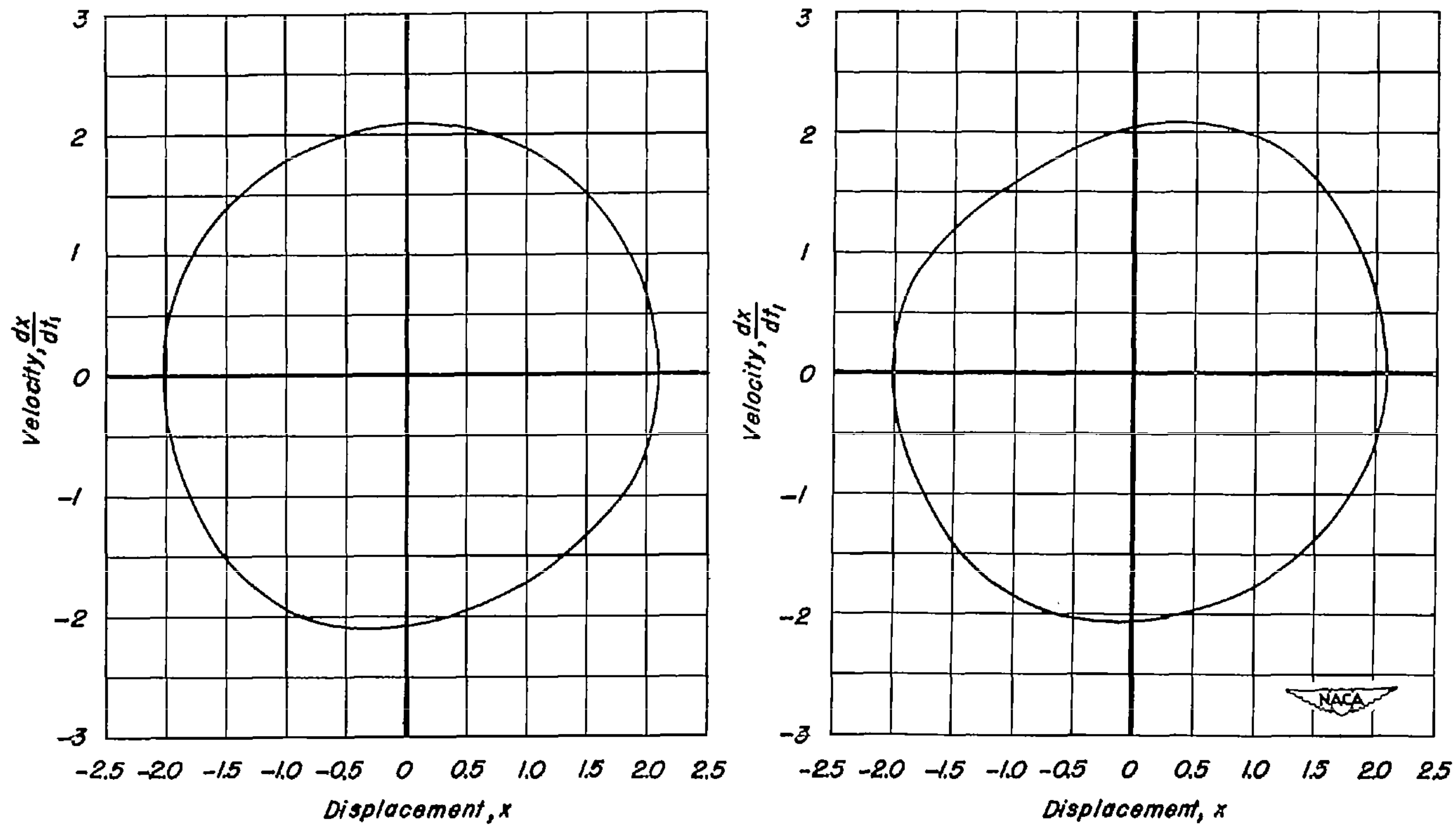
TABLE III.— CALCULATIONS FOR NONLINEAR SERVO EXAMPLE

| t | $10^3 v_f$ | $10^3 v_i$ | $10^3 v_e$ | $10^3 v_{f_s}$ | $10^3 \dot{v}_f$ | $10^2 \ddot{v}_f$ | $10 \dddot{v}_f$ | $\ddddot{v}_f$ | I |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1748 | 1493 | –243 | 1727 | 81 | –9000 | 2192 | 15020 | –27.22 |
| .025 | 1690 | 1404 | –292 | 1696 | –2111 | –8077 | 4990 | 8280 | –34.98 |
| .050 | 1605 | 1281 | –334 | 1615 | –3945 | –6619 | 6389 | 2524 | –39.56 |
| .075 | 1500 | 1126 | –369 | 1495 | –5329 | –5000 | 6300 | –2851 | –41.70 |
| .100 | 1331 | 943 | –405 | 1348 | –6326 | –3552 | 5173 | –5674 | –42.60 |
| .125 | 1180 | 737 | –447 | 1184 | –7009 | –2445 | 3685 | –5861 | –43.03 |
| .150 | 1009 | 513 | –492 | 1005 | –7516 | –1697 | 2392 | –4256 | –43.63 |
| .175 | 820 | 276 | –539 | 815 | –7915 | –1214 | 1577 | –2411 | –44.36 |
| .200 | 623 | 33 | –580 | 613 | –8237 | –881 | 1110 | –1484 | –45.09 |
| .225 | 400 | –212 | –614 | 402 | –8470 | –648 | 757 | –1457 | –45.65 |
| .250 | 184 | –451 | –638 | 187 | –8613 | –504 | 347 | –1886 | –46.04 |
| .275 | –40 | –679 | –645 | –34 | –8691 | –471 | –78 | –1111 | –46.44 |
| .300 | –247 | –890 | –645 | –245 | –8735 | –506 | –24 | 2565 | –46.74 |
| .325 | –460 | –1080 | –620 | –460 | –8784 | –429 | 658 | 4705 | –46.55 |
| .350 | –677 | –1243 | –568 | –675 | –8873 | –93 | 2430 | 9496 | –45.93 |
| .375 | –890 | –1375 | –481 | –894 | –8831 | 857 | 5181 | 11510 | –41.93 |
| .400 | –1098 | –1474 | –364 | –1110 | –8535 | 2490 | 7830 | 9273 | –34.15 |
| .425 | –1300 | –1535 | –218 | –1317 | –7748 | 4692 | 9512 | 3460 | –22.77 |
| .450 | –1502 | –1560 | –58 | –1502 | –6324 | 7098 | 9328 | –5161 | –7.97 |
| .475 | –1640 | –1546 | 95 | –1641 | –4244 | 9183 | 6945 | –13610 | 8.65 |
| .500 | –1748 | –1493 | 228 | –1721 | –1636 | 10432 | 2798 | –18770 | 24.94 |
| .525 | –1748 | –1404 | 322 | –1726 | 1106 | 10529 | –2012 | –18840 | 38.17 |
| .550 | –1659 | –1282 | 377 | –1659 | 3710 | 9473 | –6218 | –14100 | 47.25 |
| .575 | –1520 | –1126 | 411 | –1536 | 5868 | 7546 | –8827 | –6661 | 51.75 |
| .600 | –1349 | –943 | 417 | –1360 | 7413 | 5220 | –9435 | 2212 | 52.33 |
| .625 | –1150 | –737 | 425 | –1162 | 8351 | 2987 | –7954 | 7838 | 50.59 |
| .650 | –959 | –513 | 437 | –950 | 8793 | 1307 | –5828 | 9637 | 48.16 |
| .675 | –730 | –276 | 454 | –730 | 8715 | 106 | –3262 | 10110 | 45.67 |
| .700 | –507 | –33 | 476 | –509 | 8869 | –343 | –971 | 7998 | 44.60 |
| .725 | –290 | 212 | 500 | –288 | 8791 | –260 | 731 | 6056 | 44.89 |
| .750 | –67 | 451 | 519 | –68 | 8764 | 130 | 1833 | 1284 | 46.21 |
| .775 | 160 | 679 | 528 | 151 | 8826 | 534 | 1274 | –4718 | 47.60 |
| .800 | 372 | 890 | 519 | 371 | 8991 | 677 | –346 | –8389 | 48.62 |
| .825 | 590 | 1080 | 485 | 595 | 9132 | 295 | –2805 | –10830 | 47.31 |
| .850 | 807 | 1243 | 420 | 823 | 9155 | –744 | –5516 | –13260 | 43.42 |
| .875 | 1040 | 1375 | 324 | 1051 | 8833 | –2419 | –7692 | –6966 | 35.98 |
| .900 | 1273 | 1474 | 205 | 1269 | 8024 | –4487 | –8644 | 839 | 25.12 |
| .925 | 1460 | 1535 | 77 | 1458 | 6638 | –6573 | –7384 | 7186 | 11.76 |
| .950 | 1614 | 1560 | –45 | 1605 | 4725 | –8044 | –5291 | 10350 | –2.18 |
| .975 | 1700 | 1546 | –151 | 1697 | 2448 | –9077 | –2048 | 16030 | –16.32 |

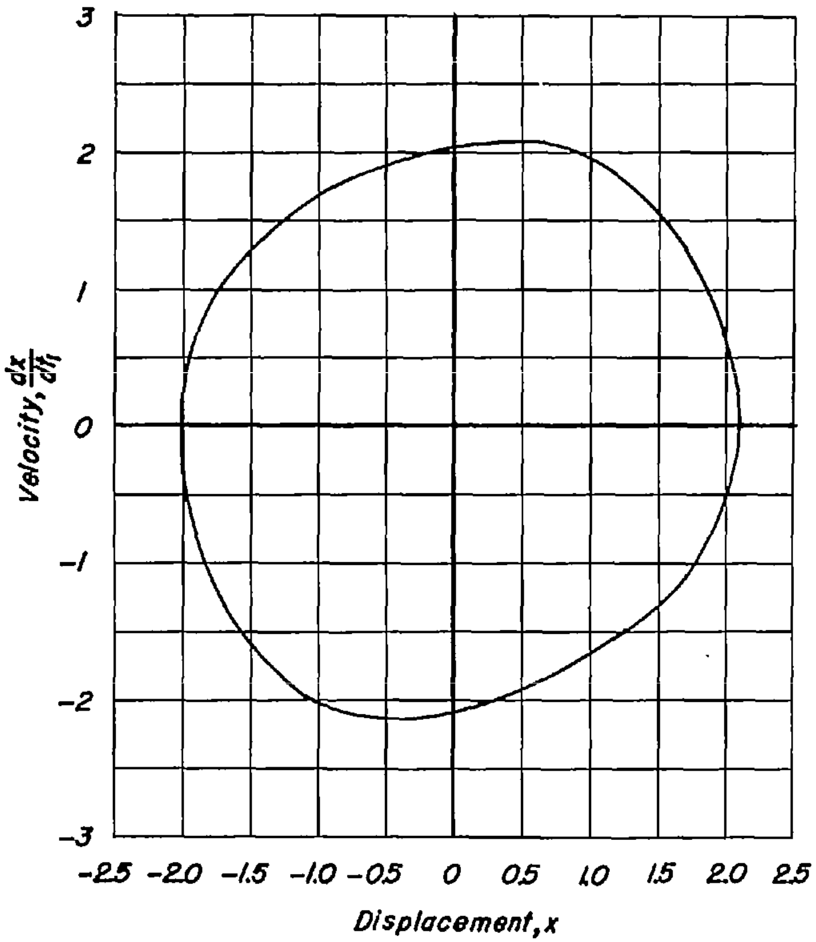
TABLE IV.– CALCULATIONS FOR SELF-SUSTAINED OSCILLATION EXAMPLE

| t | $10^8\psi$ | $10^4\psi_s$ | $10^3\dot{\psi}$ | t | $10^8\psi$ | $10^4\psi_s$ | $10^3\dot{\psi}$ | t | $10^8\psi$ | $10^4\psi_s$ | $10^3\dot{\psi}$ | t | $10^8\psi$ | $10^4\psi_s$ | $10^3\dot{\psi}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | −384 | −3856 | −2090 | 0.225 | −413 | −4111 | −1888 | 0.450 | −362 | −3580 | 838 | 0.675 | −407 | −4149 | 3073 |
| .005 | −396 | −3919 | −578 | .230 | −418 | −4159 | −200 | .455 | −351 | −3505 | 2057 | .680 | −396 | −3967 | 4205 |
| .010 | −389 | −3915 | 748 | .235 | −418 | −4131 | 1478 | .460 | −333 | −3377 | 3127 | .685 | −373 | −3727 | 5472 |
| .015 | −378 | −3845 | 2025 | .240 | −396 | −4012 | 3115 | .465 | −318 | −3195 | 4008 | .690 | −340 | −3418 | 6872 |
| .020 | −373 | −3712 | 3303 | .245 | −378 | −3822 | 4557 | .470 | −296 | −2976 | 4897 | .695 | −307 | −3040 | 8258 |
| .025 | −351 | −3510 | 4903 | .250 | −362 | −3556 | 6053 | .475 | −273 | −2702 | 6035 | .700 | −267 | −2594 | 9552 |
| .030 | −329 | −3218 | 6840 | .255 | −313 | −3217 | 7498 | .480 | −240 | −2371 | 7247 | .705 | −200 | −2090 | 10517 |
| .035 | −282 | −2834 | 8172 | .260 | −284 | −2804 | 9120 | .485 | −196 | −1976 | 8425 | .710 | −156 | −1549 | 11148 |
| .040 | −240 | −2408 | 9400 | .265 | −236 | −2305 | 10755 | .490 | −151 | −1532 | 9350 | .715 | −96 | −978 | 11527 |
| .045 | −184 | −1903 | 10383 | .270 | −171 | −1732 | 12228 | .495 | −107 | −1045 | 10032 | .720 | −40 | −397 | 11788 |
| .050 | −140 | −1380 | 10743 | .275 | −111 | −1094 | 12907 | .500 | −49 | −533 | 10465 | .725 | −16 | 200 | 12035 |
| .055 | −87 | −840 | 10945 | .280 | −40 | −453 | 12797 | .505 | 7 | −3 | 10645 | .730 | 82 | 802 | 11970 |
| .060 | −29 | −290 | 10930 | .285 | −16 | 183 | 12585 | .510 | 56 | 523 | 10432 | .735 | 138 | 1393 | 11697 |
| .065 | 27 | 250 | 10748 | .290 | 82 | 796 | 11703 | .515 | 104 | 1032 | 9667 | .740 | 200 | 1966 | 11078 |
| .070 | 82 | 782 | 10392 | .295 | 133 | 1347 | 10513 | .520 | 149 | 1488 | 8683 | .745 | 249 | 2495 | 10093 |
| .075 | 127 | 1287 | 9868 | .300 | 184 | 1847 | 9367 | .525 | 193 | 1901 | 7807 | .750 | 293 | 2971 | 8822 |
| .080 | 178 | 1771 | 9550 | .305 | 227 | 2283 | 8187 | .530 | 227 | 2267 | 6770 | .755 | 340 | 3377 | 7613 |
| .085 | 216 | 2241 | 9170 | .310 | 271 | 2664 | 6888 | .535 | 249 | 2580 | 5910 | .760 | 378 | 3729 | 6167 |
| .090 | 271 | 2684 | 8552 | .315 | 298 | 2971 | 5512 | .540 | 293 | 2858 | 5057 | .765 | 396 | 3981 | 4537 |
| .095 | 311 | 3089 | 7475 | .320 | 316 | 3218 | 4390 | .545 | 304 | 3083 | 4017 | .770 | 416 | 4173 | 2978 |
| .100 | 338 | 3428 | 5947 | .325 | 338 | 3408 | 3080 | .550 | 327 | 3257 | 2835 | .775 | 427 | 4277 | 1273 |
| .105 | 376 | 3676 | 3852 | .330 | 353 | 3526 | 1882 | .555 | 338 | 3362 | 1351 | .780 | 427 | 4300 | −425 |
| .110 | 382 | 3812 | 1978 | .335 | 360 | 3587 | 0 | .560 | 342 | 3389 | −352 | .785 | 427 | 4233 | −2222 |
| .115 | 387 | 3882 | 847 | .340 | 360 | 3521 | −2340 | .565 | 327 | 3328 | −1807 | .790 | 409 | 4076 | −4155 |
| .120 | 382 | 3885 | −1360 | .345 | 333 | 3368 | −3458 | .570 | 318 | 3206 | −3313 | .795 | 382 | 3818 | −6062 |
| .125 | 378 | 3752 | −3257 | .350 | 311 | 3178 | −4473 | .575 | 304 | 2992 | −5180 | .800 | 342 | 3473 | −7748 |
| .130 | 360 | 3570 | −4203 | .355 | 291 | 2918 | −5772 | .580 | 271 | 2689 | −6935 | .805 | 311 | 3046 | −9283 |
| .135 | 327 | 3326 | −5605 | .360 | 271 | 2605 | −6827 | .585 | 233 | 2300 | −8655 | .810 | 249 | 2549 | −10540 |
| .140 | 298 | 2999 | −6888 | .365 | 220 | 2237 | −7843 | .590 | 178 | 1831 | −9882 | .815 | 204 | 1996 | −11553 |
| .145 | 271 | 2631 | −7658 | .370 | 182 | 1826 | −8388 | .595 | 138 | 1321 | −10500 | .820 | 138 | 1397 | −12400 |
| .150 | 216 | 2227 | −8902 | .375 | 138 | 1402 | −8685 | .600 | 71 | 786 | −10848 | .825 | 78 | 762 | −12857 |
| .155 | 171 | 1745 | −9825 | .380 | 93 | 958 | −8995 | .605 | 22 | 240 | −10947 | .830 | 4 | 117 | −12900 |
| .160 | 127 | 1252 | −10202 | .385 | 49 | 504 | −9197 | .610 | −29 | −308 | −11020 | .835 | −40 | −530 | −13250 |
| .165 | 76 | 726 | −10762 | .390 | 18 | 39 | −9388 | .615 | −84 | −863 | −11195 | .840 | −129 | −1169 | −13010 |
| .170 | 16 | 187 | −10995 | .395 | −51 | −432 | −9412 | .620 | −140 | −1426 | −11317 | .845 | −173 | −1783 | −12020 |
| .175 | −40 | −370 | −11052 | .400 | −95 | −897 | −9075 | .625 | −196 | −1987 | −10898 | .850 | −238 | −2364 | −11212 |
| .180 | −84 | −919 | −10842 | .405 | −133 | −1337 | −8600 | .630 | −251 | −2510 | −10165 | .855 | −284 | −2901 | −10247 |
| .185 | −151 | −1450 | −10378 | .410 | −173 | −1756 | −8062 | .635 | −307 | −2996 | −8952 | .860 | −347 | −3383 | −8895 |
| .190 | −196 | −1953 | −9688 | .415 | −212 | −2144 | −7593 | .640 | −340 | −3390 | −7312 | .865 | −373 | −3787 | −7328 |
| .195 | −240 | −2415 | −8745 | .420 | −253 | −2516 | −7262 | .645 | −373 | −3728 | −5798 | .870 | −418 | −4115 | −5742 |
| .200 | −284 | −2826 | −7647 | .425 | −284 | −2861 | −6208 | .650 | −396 | −3979 | −4260 | .875 | −429 | −4360 | −4072 |
| .205 | −318 | −3187 | −6978 | .430 | −313 | −3135 | −5003 | .655 | −407 | −4157 | −2968 | .880 | −451 | −4522 | −2418 |
| .210 | −351 | −3531 | −5848 | .435 | −340 | −3361 | −3847 | .660 | −429 | −4275 | −1627 | .885 | −462 | −4600 | −665 |
| .215 | −373 | −3768 | −4507 | .440 | −351 | −3514 | −2245 | .665 | −440 | −4315 | 40 | | | | |
| .220 | −396 | −3972 | −3532 | .445 | −356 | −3585 | −640 | .670 | −429 | −4271 | 1707 | | | | |

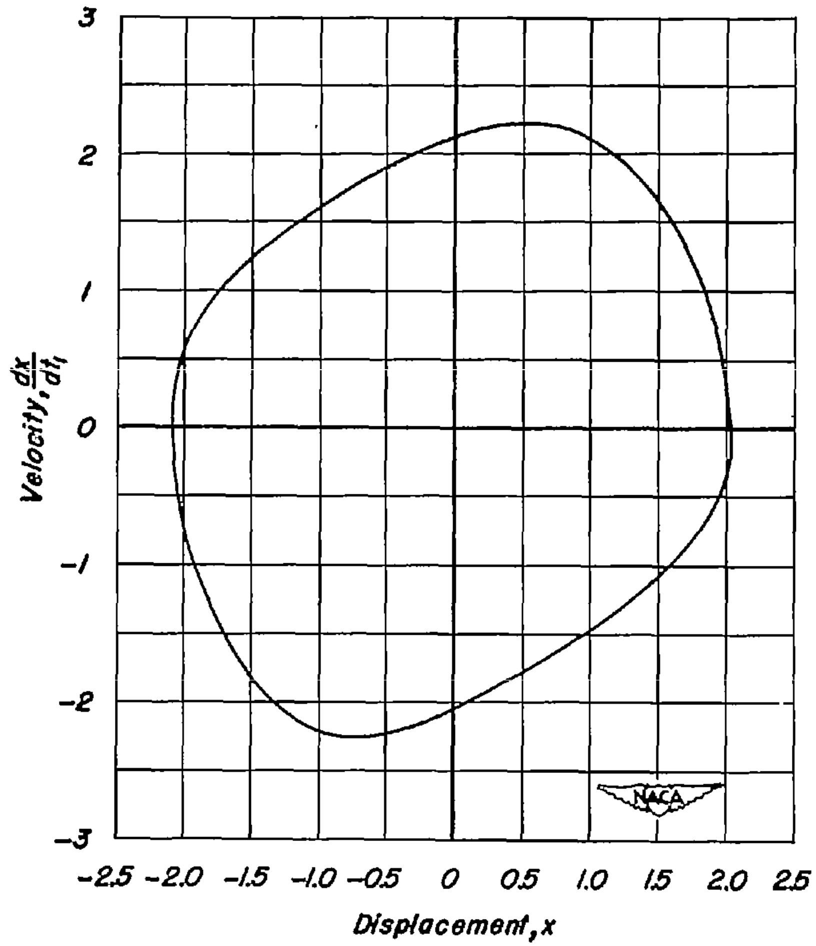



(a) $\mu = 0.1$ (b) $\mu = 0.16$

Figure 1.– Limit cycles of Van der Pol's equation for various values of $\mu$.

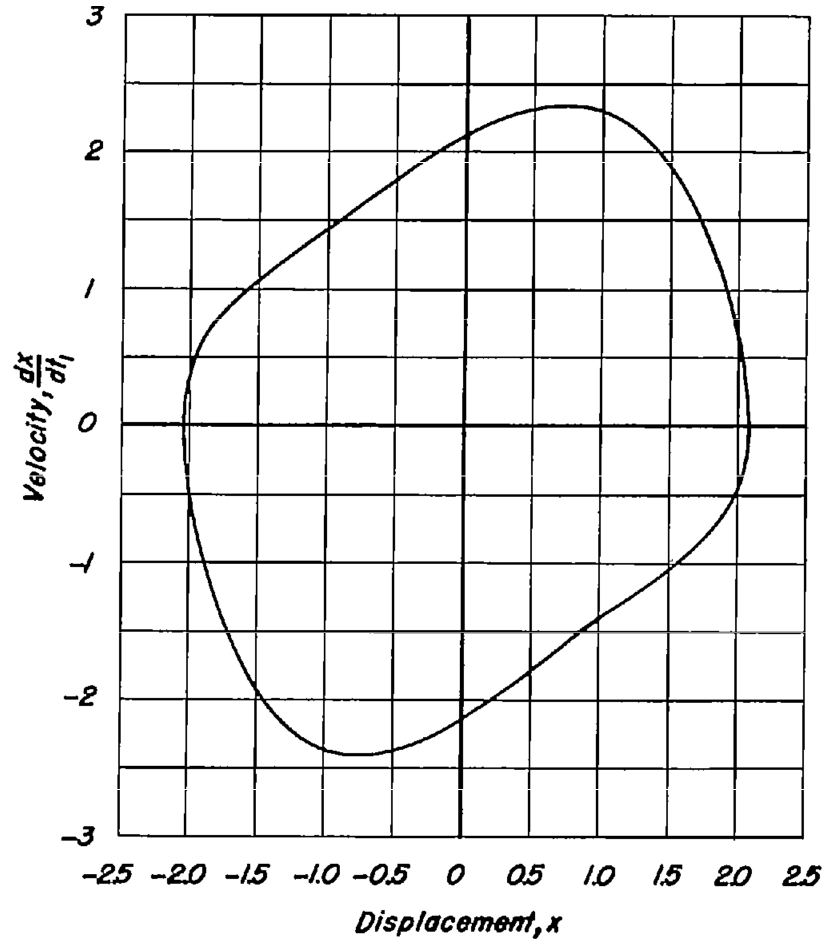



(c) $\mu = 0.2$

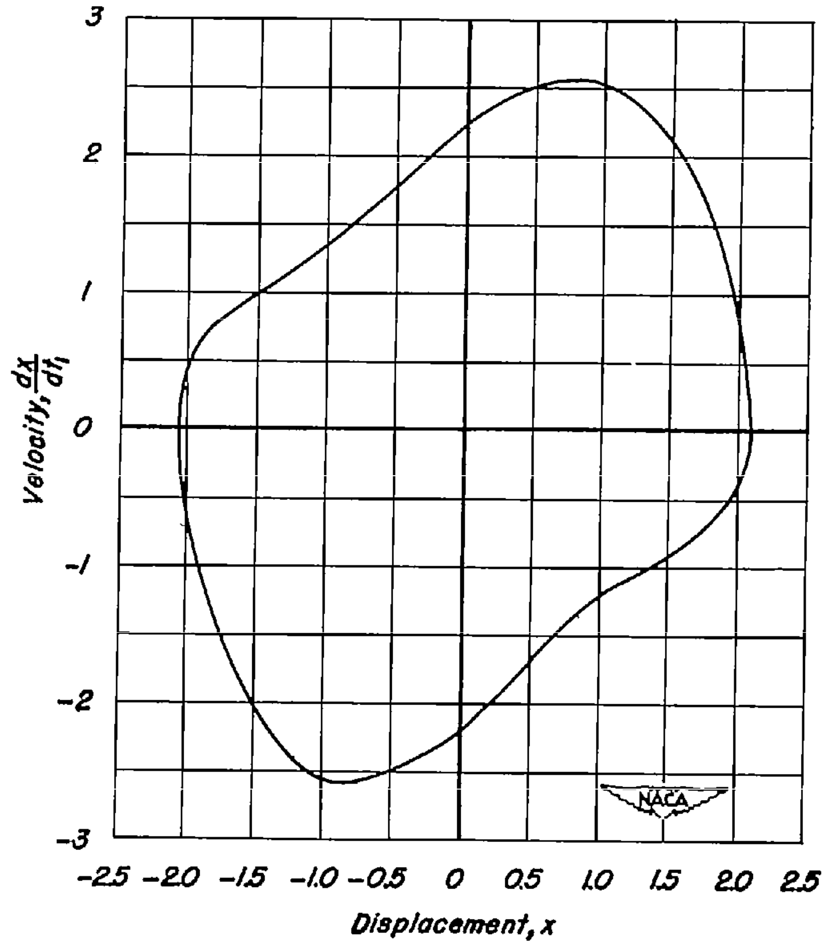



(d) $\mu = 0.4$

Figure 1.– Continued.

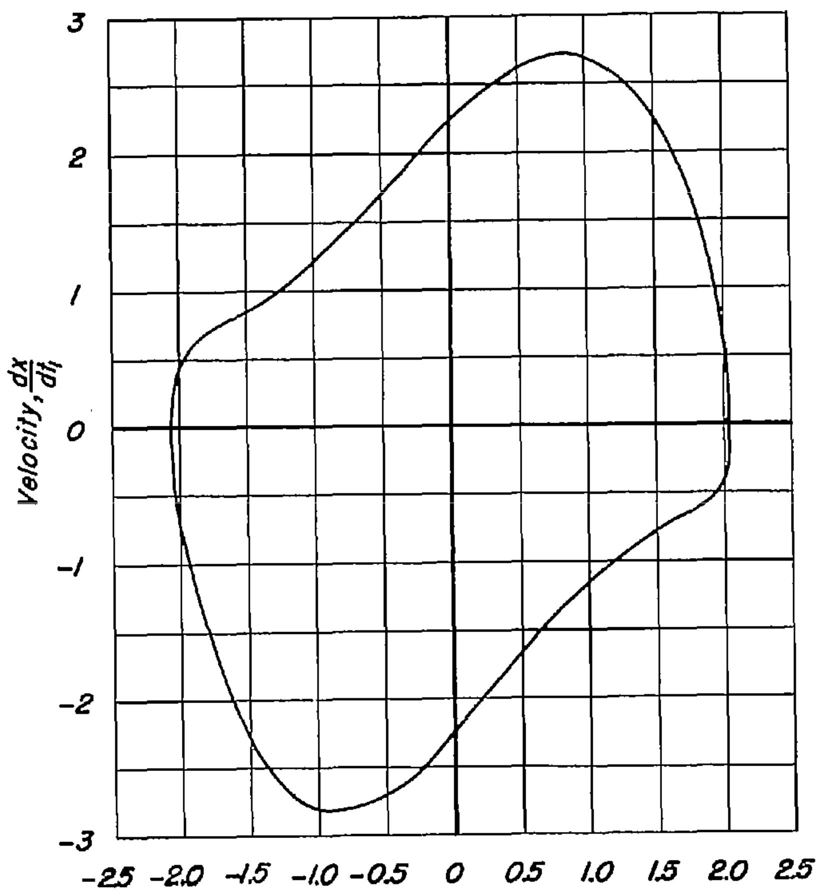



(e) $\mu = 0.6$

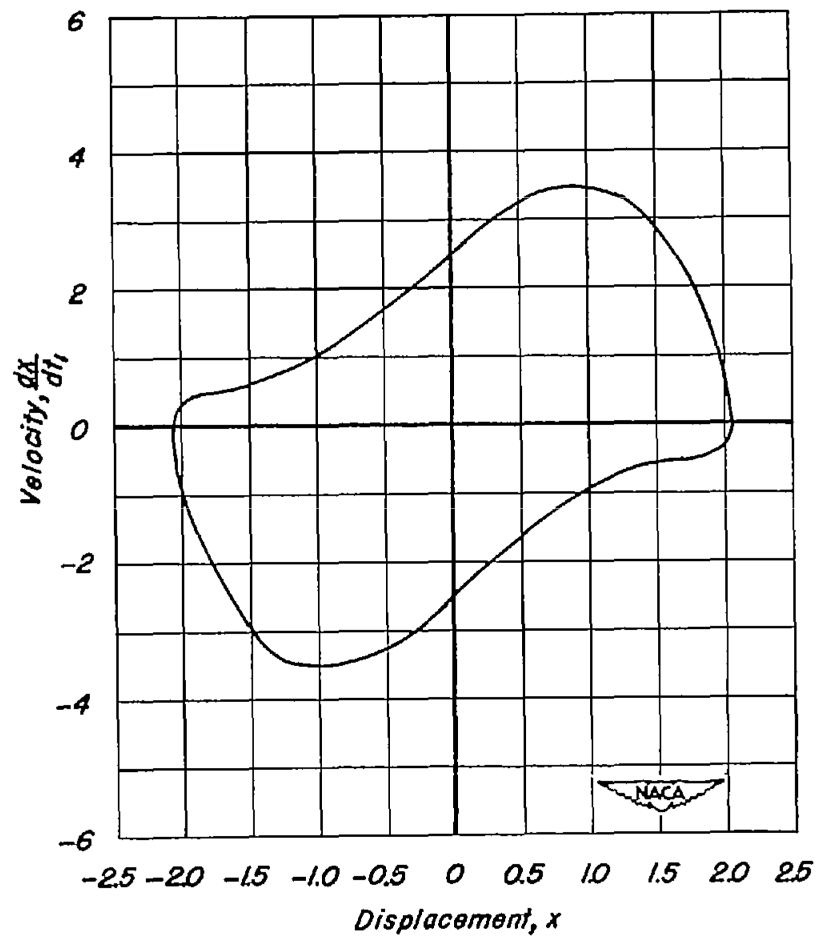



(f) $\mu = 0.8$

Figure 1.– Continued.

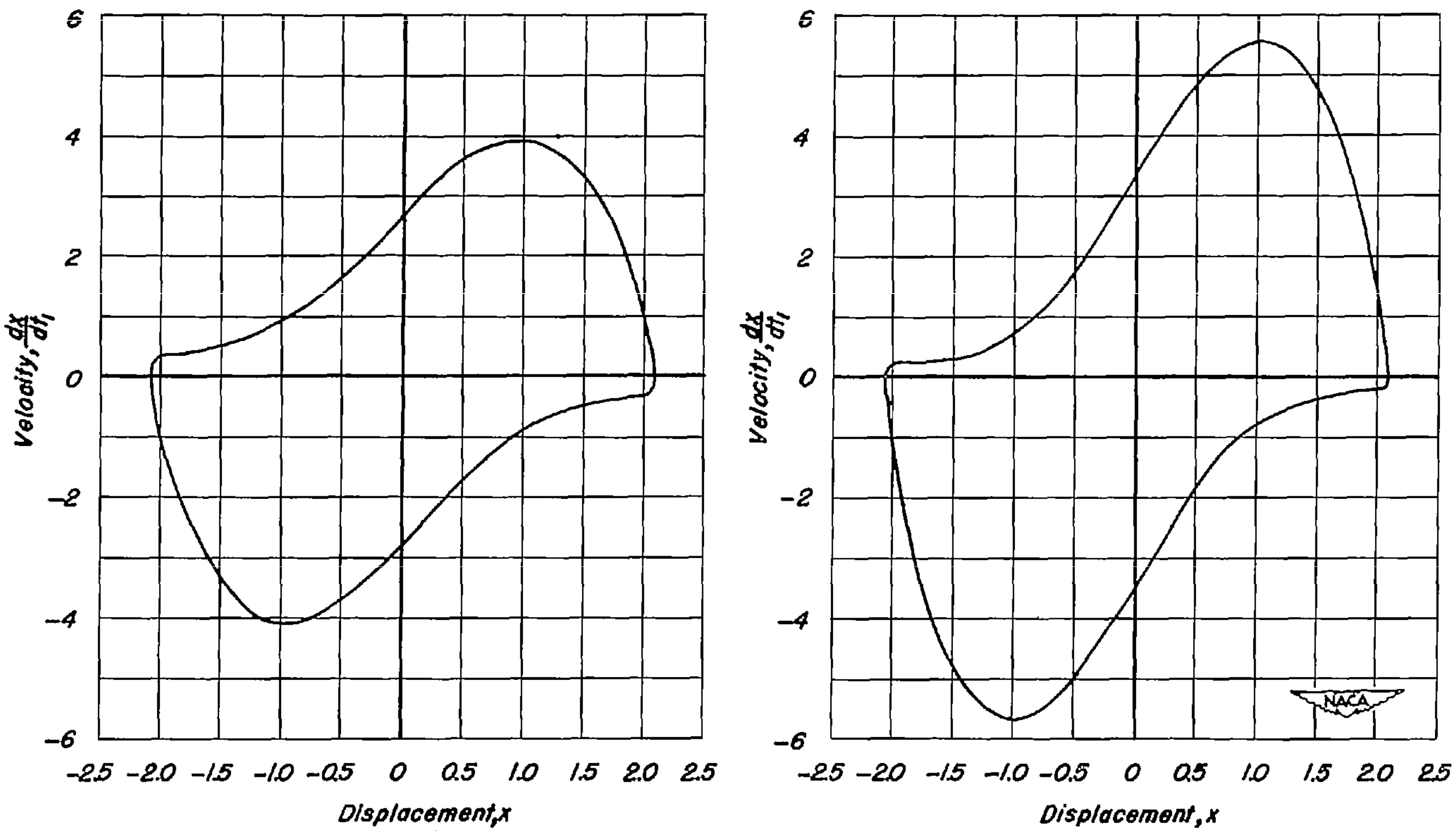



(g) $\mu = 1.0$




(h) $\mu = 1.6$

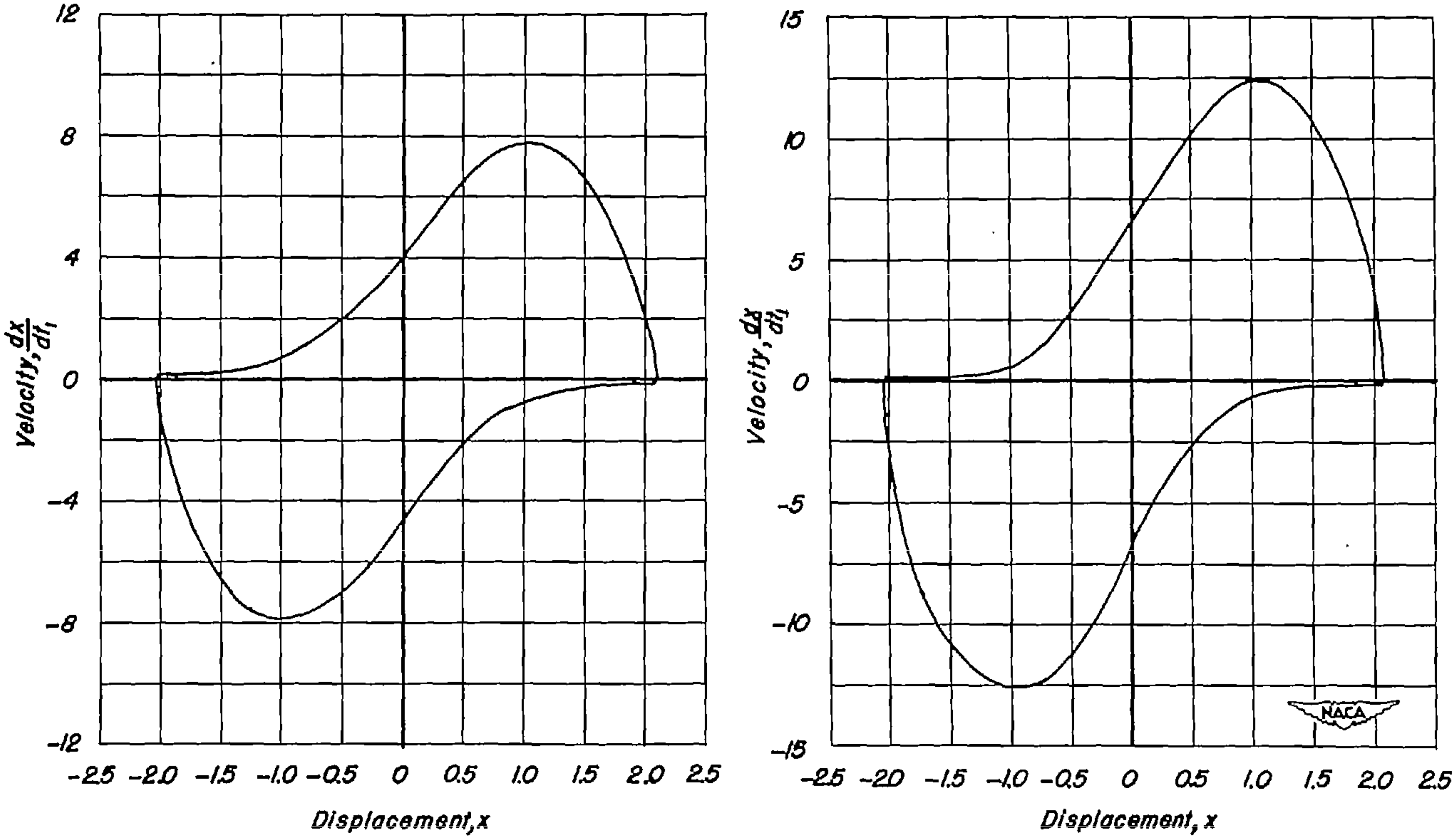
Figure 1.– Continued.




(i) $\mu = 2.0$

(j) $\mu = 3.2$

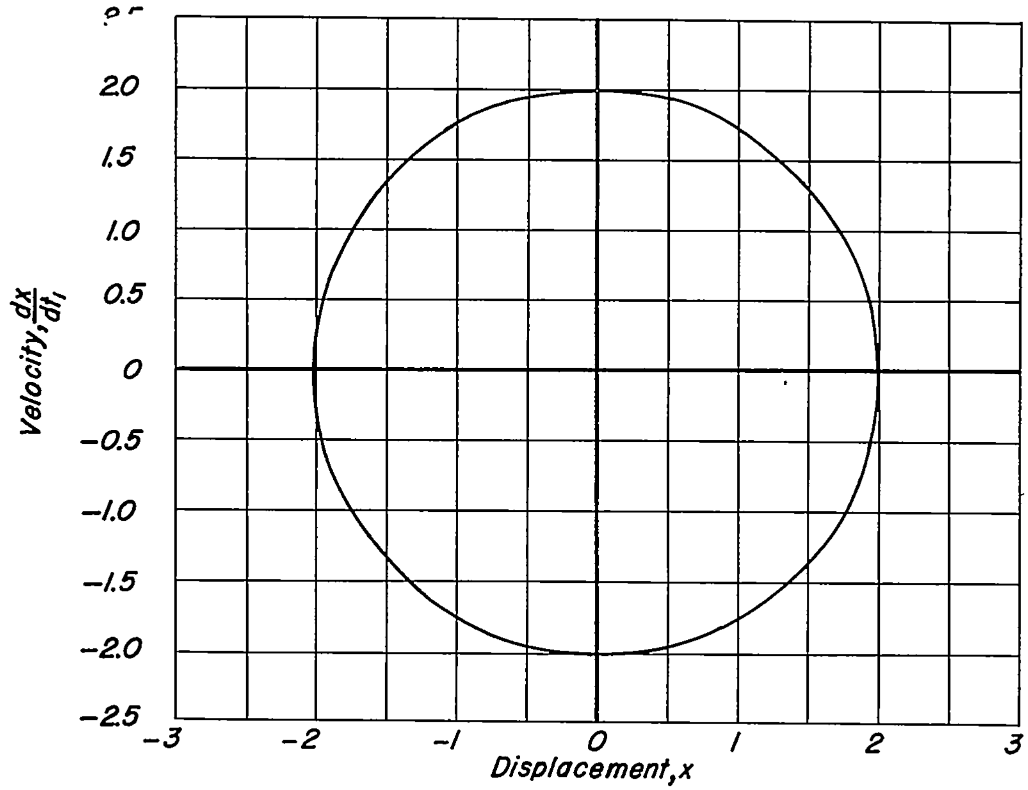
Figure 1.– Continued.



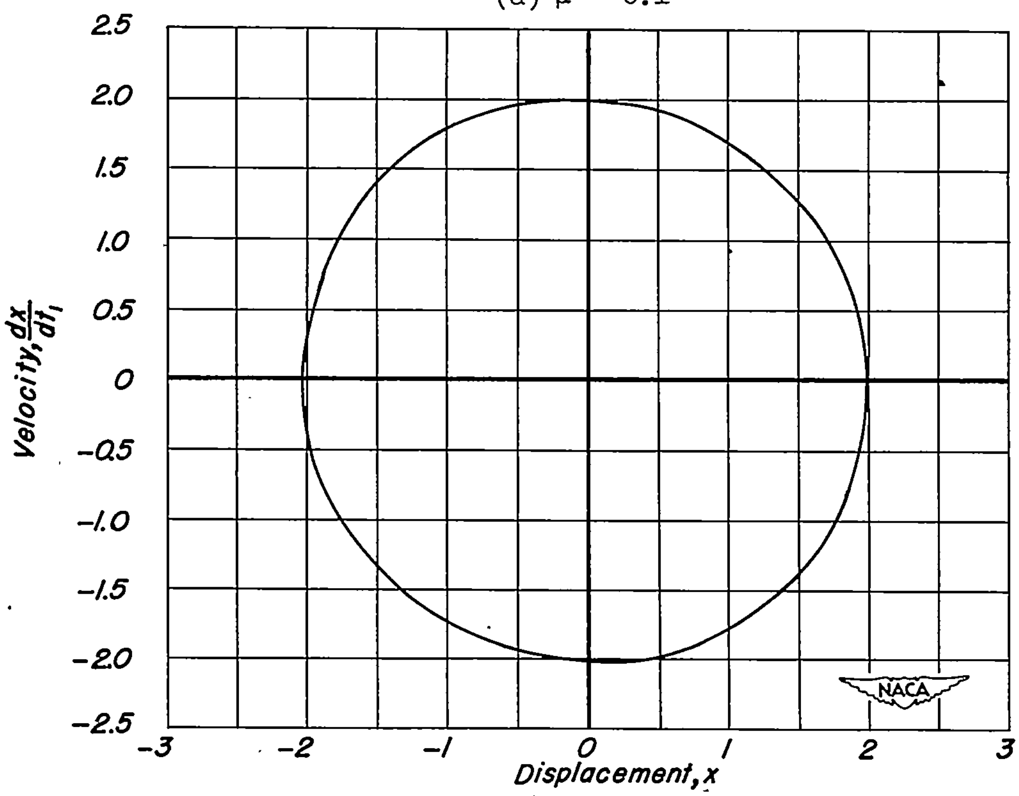

(k) $\mu = 4.8$

(l) $\mu = 8.0$

Figure 1.– Concluded.

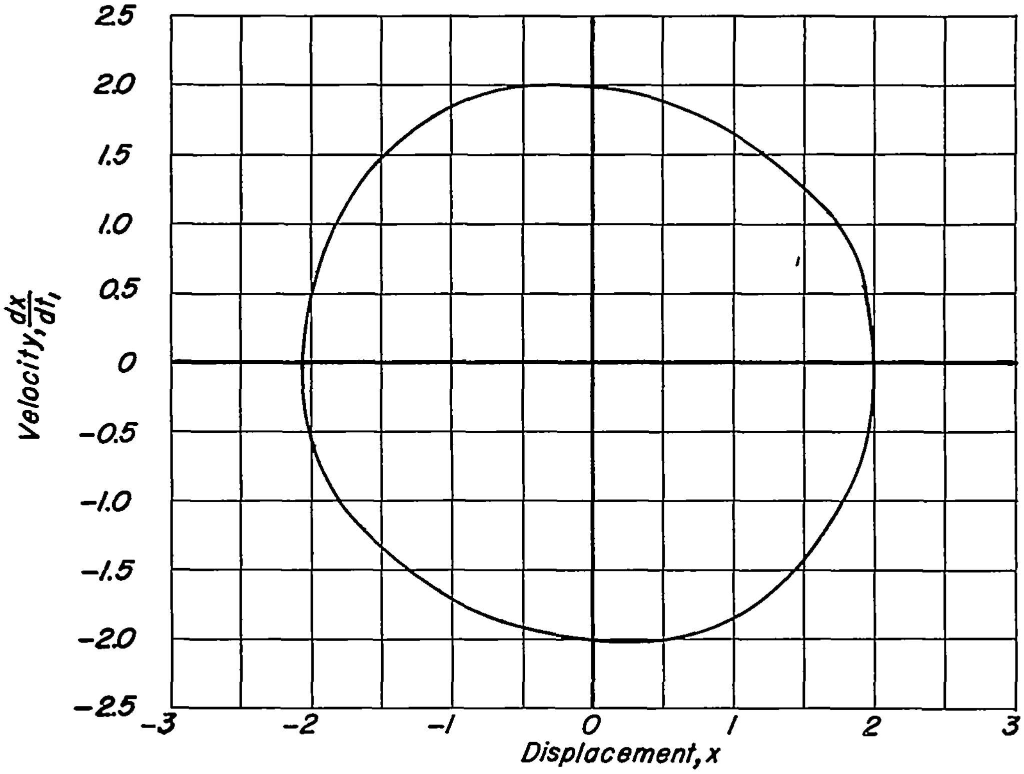



(a) $\mu = 0.1$

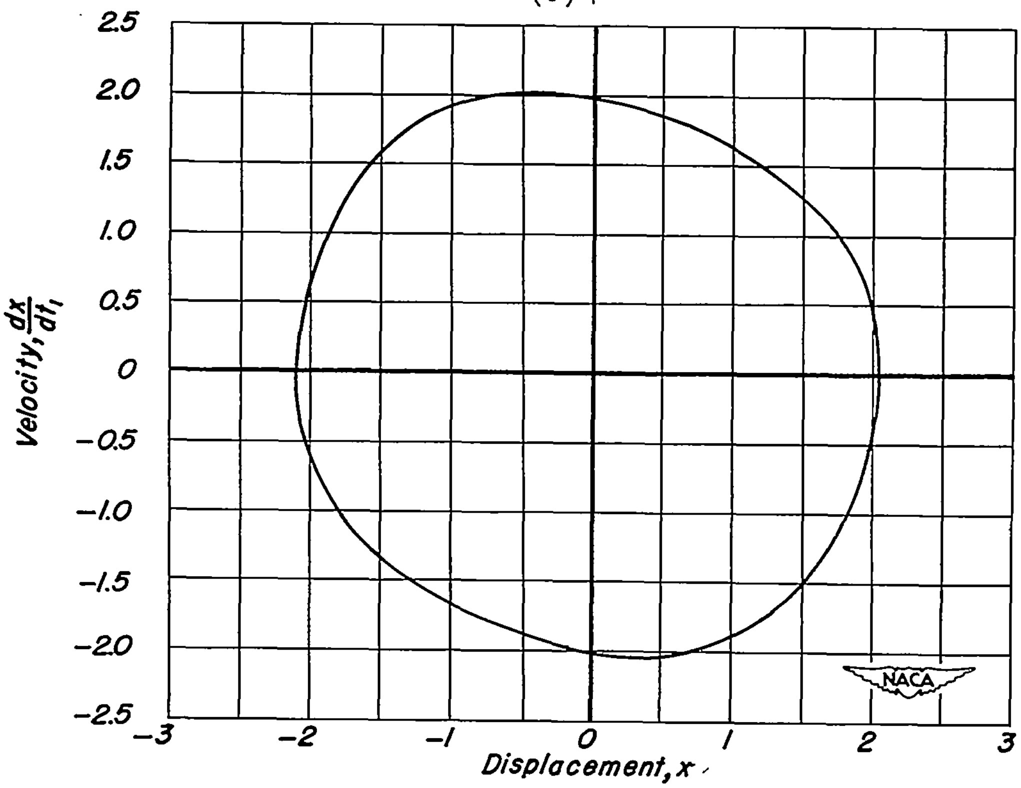



(b) $\mu = 0.2$

Figure 2.— Limit cycles of Rayleigh's equation for various values of $\mu$.

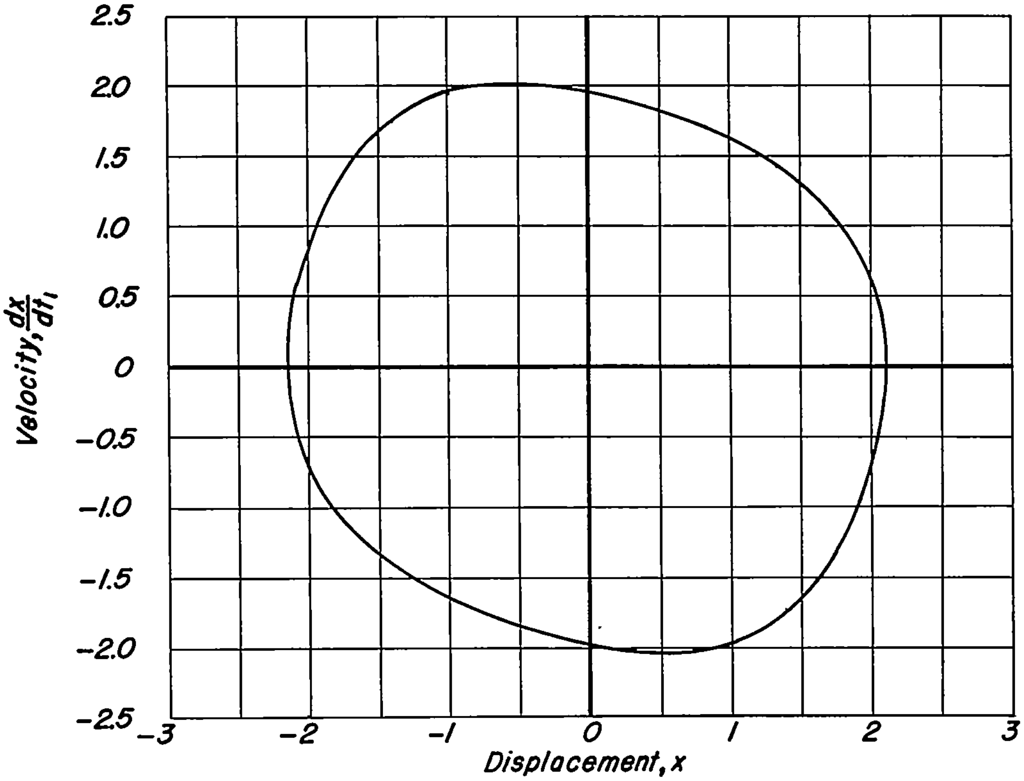



(c) $\mu = 0.4$

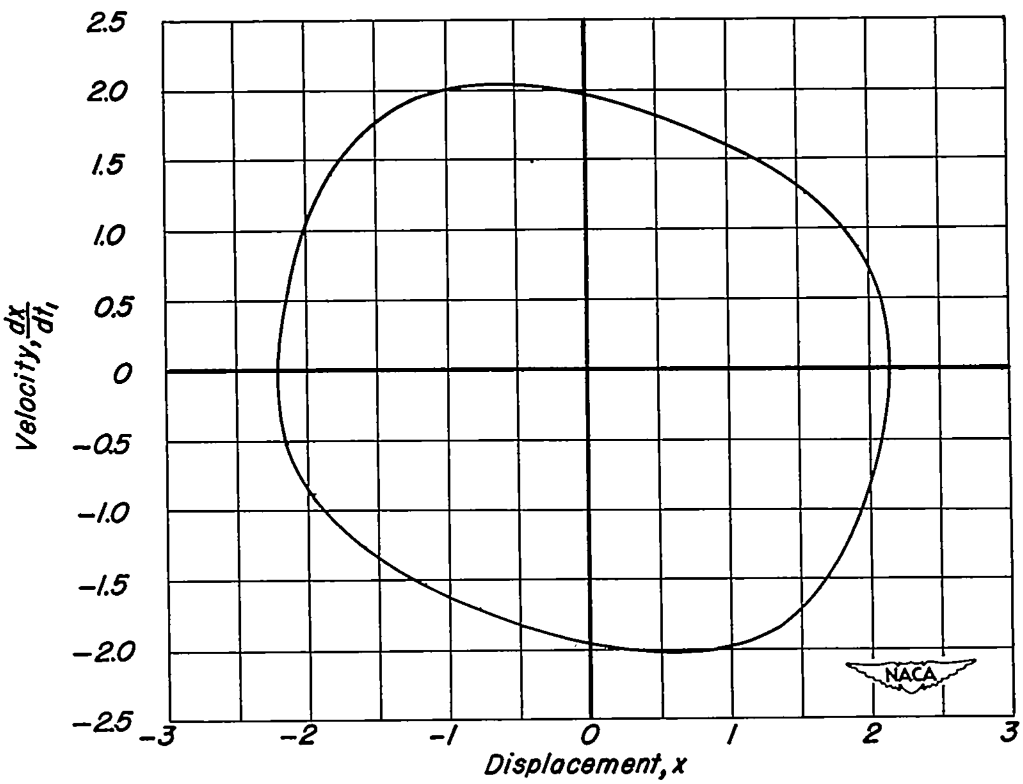



(d) $\mu = 0.6$

Figure 2.- Continued.

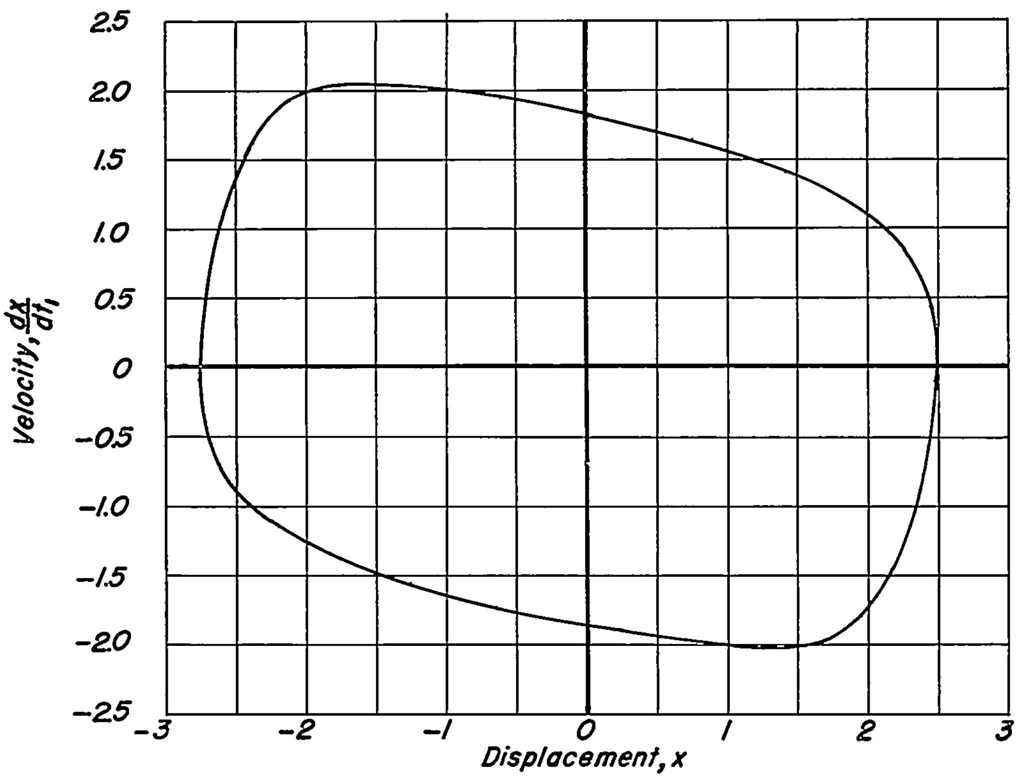



(e) $\mu$ = 0.8

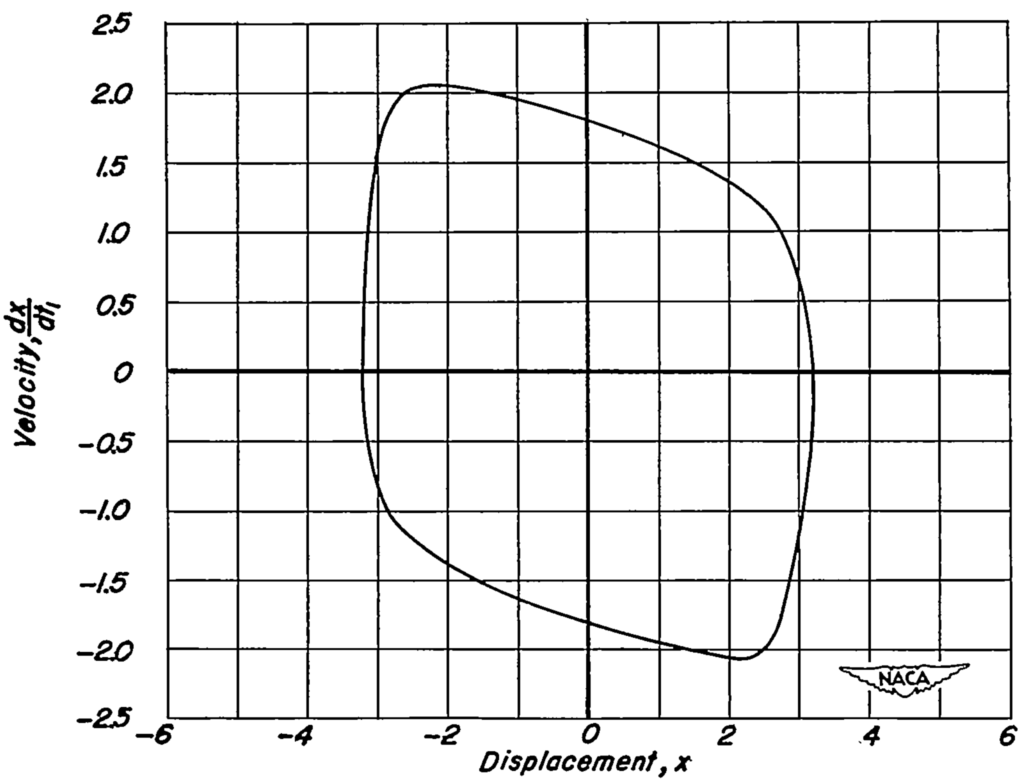



(f) $\mu$ = 1.0

Figure 2.- Continued.

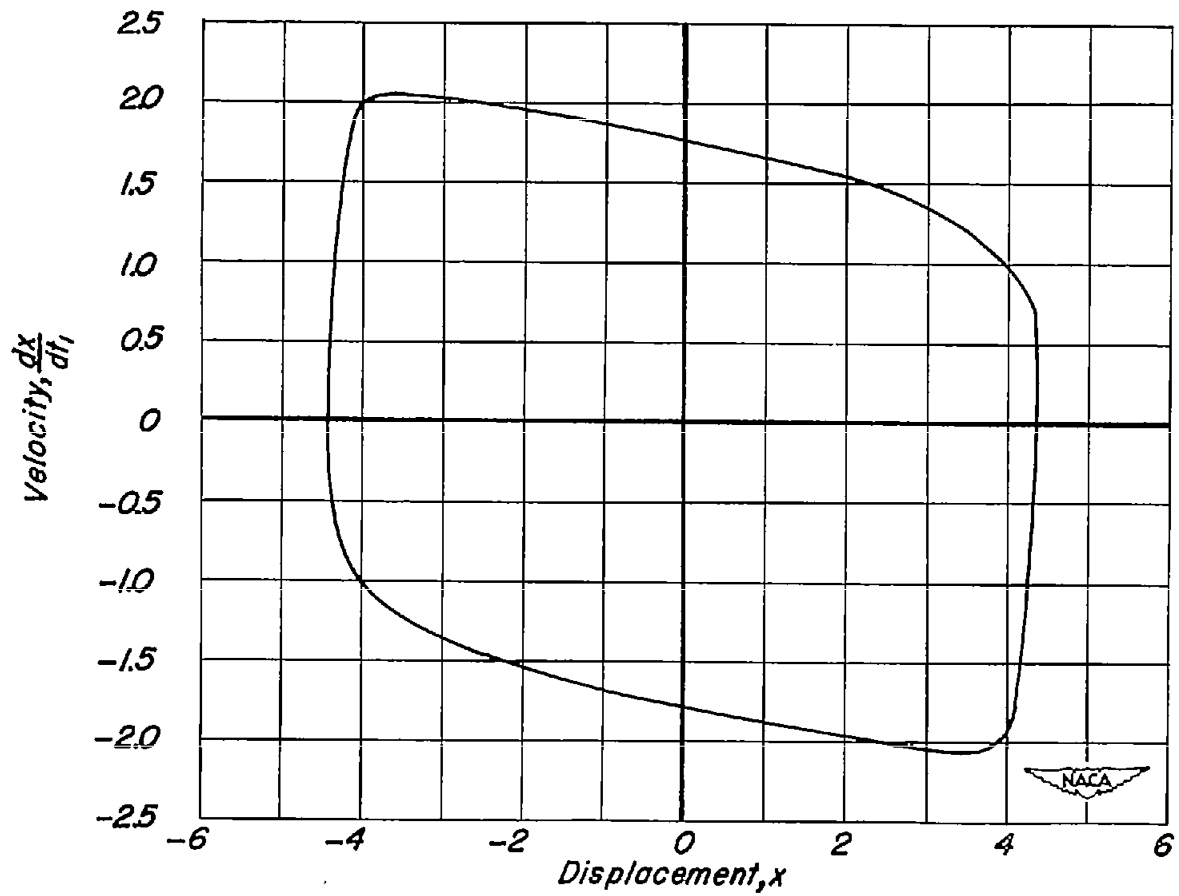



(g) $\mu = 2.0$




(h) $\mu = 3.0$

Figure 2.- Continued.




(i) $\mu = 5$

Figure 2.– Concluded.

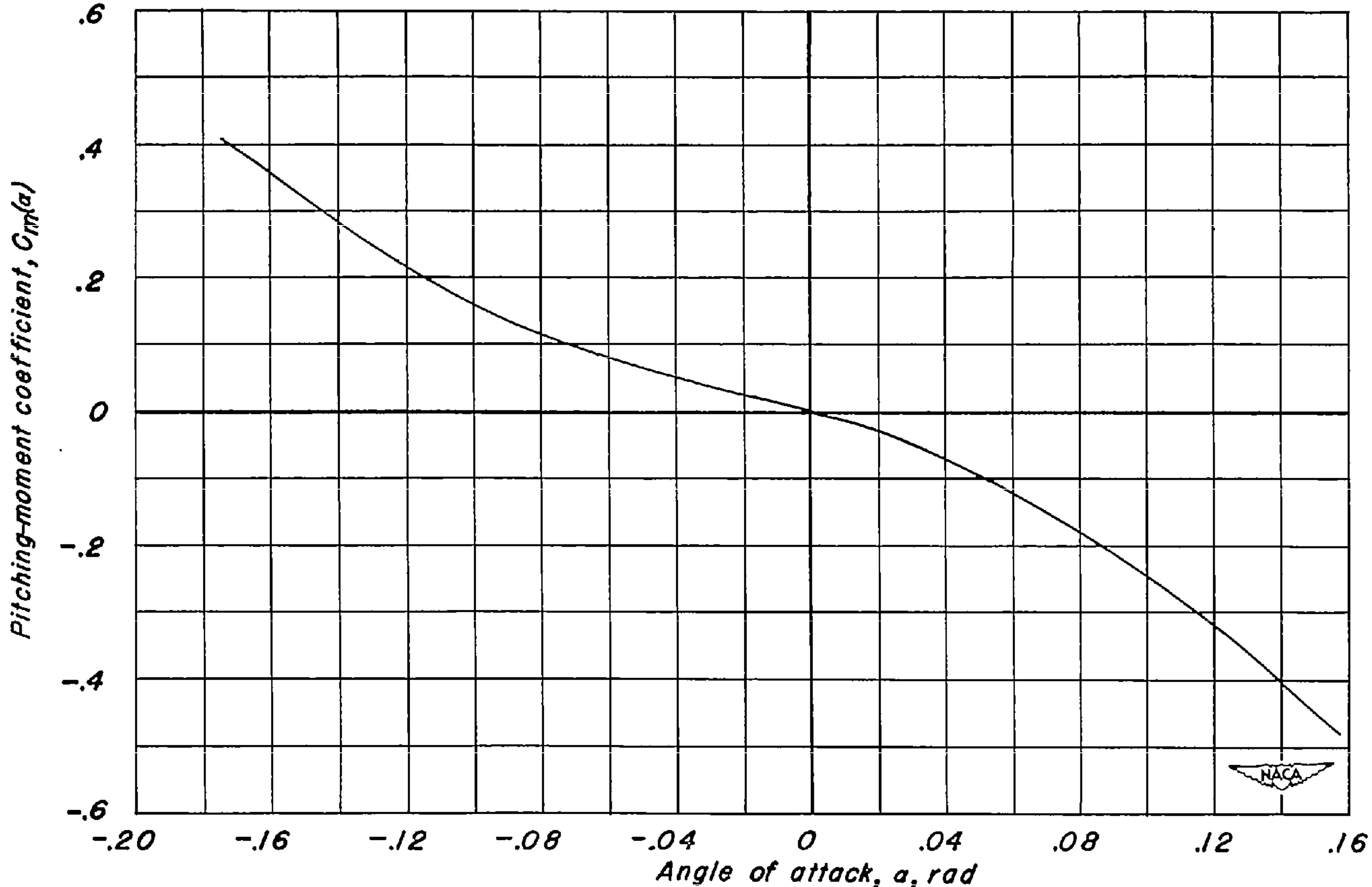



Figure 3.— Nonlinear variation of pitching-moment coefficient with angle of attack from wind-tunnel tests of a missile.

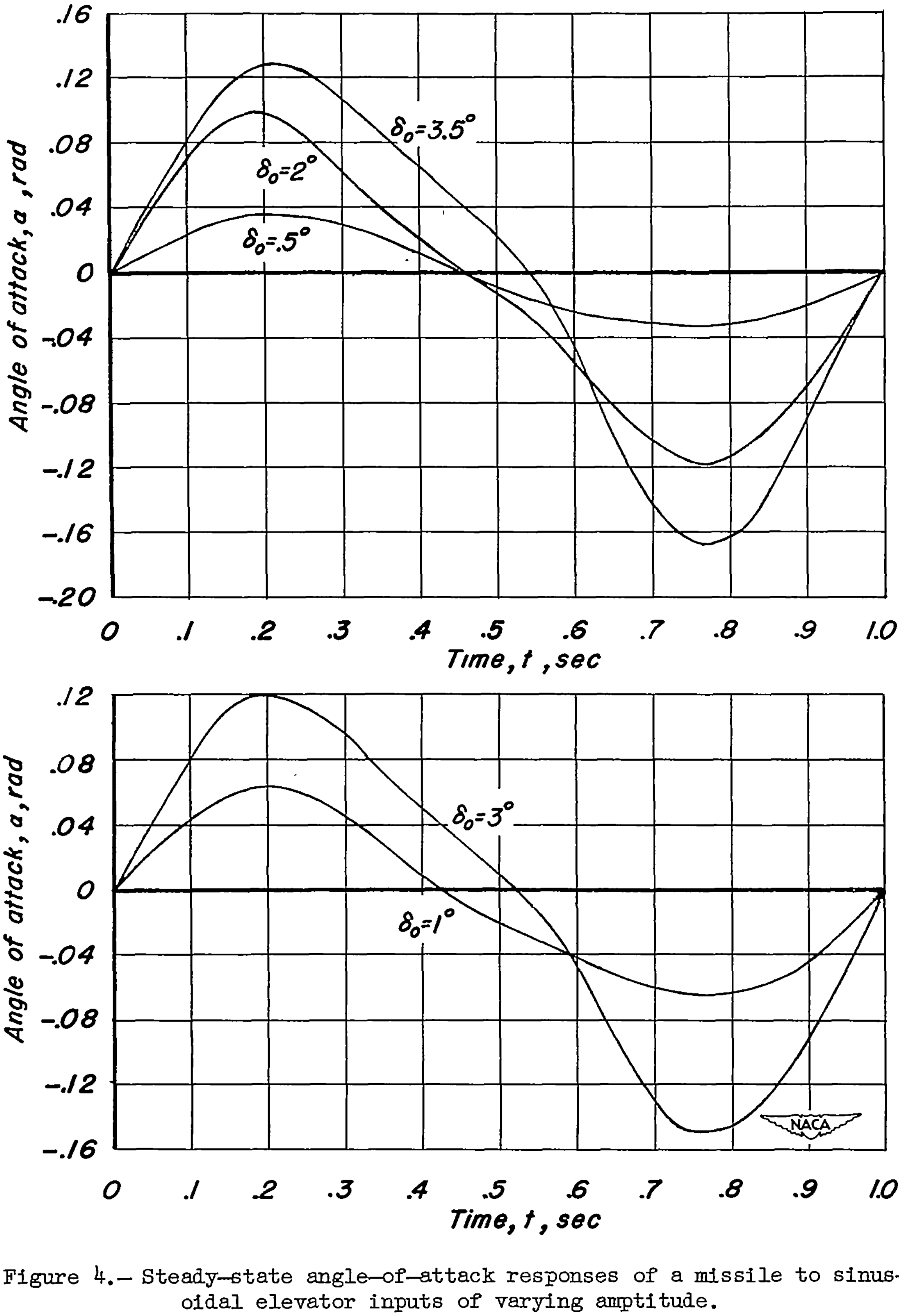



Figure 4.- Steady-state angle-of-attack responses of a missile to sinusoidal elevator inputs of varying amptitude.

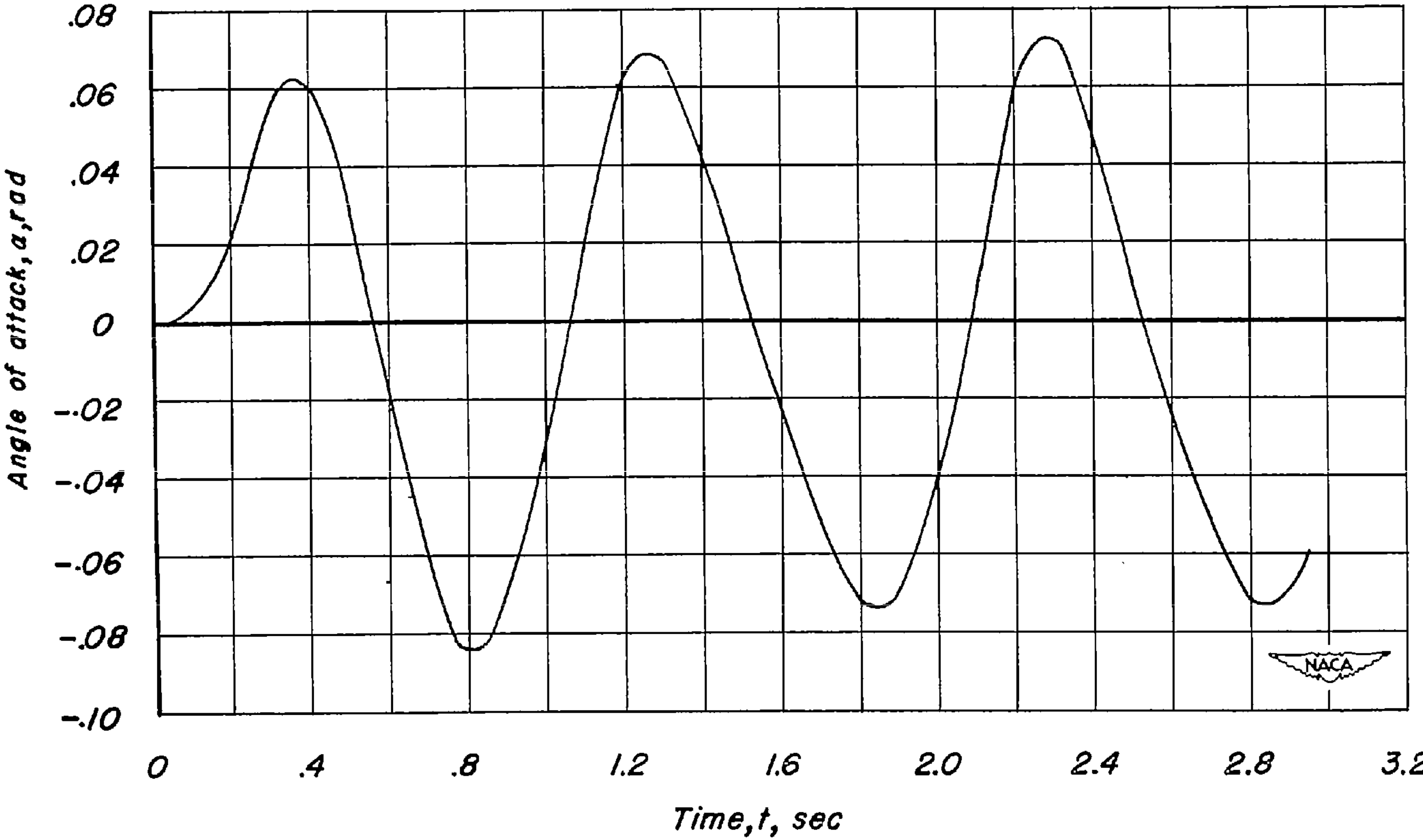



Figure 5.- Angle-of-attack response of a missile to a sinusoidal elevator input of 1°.

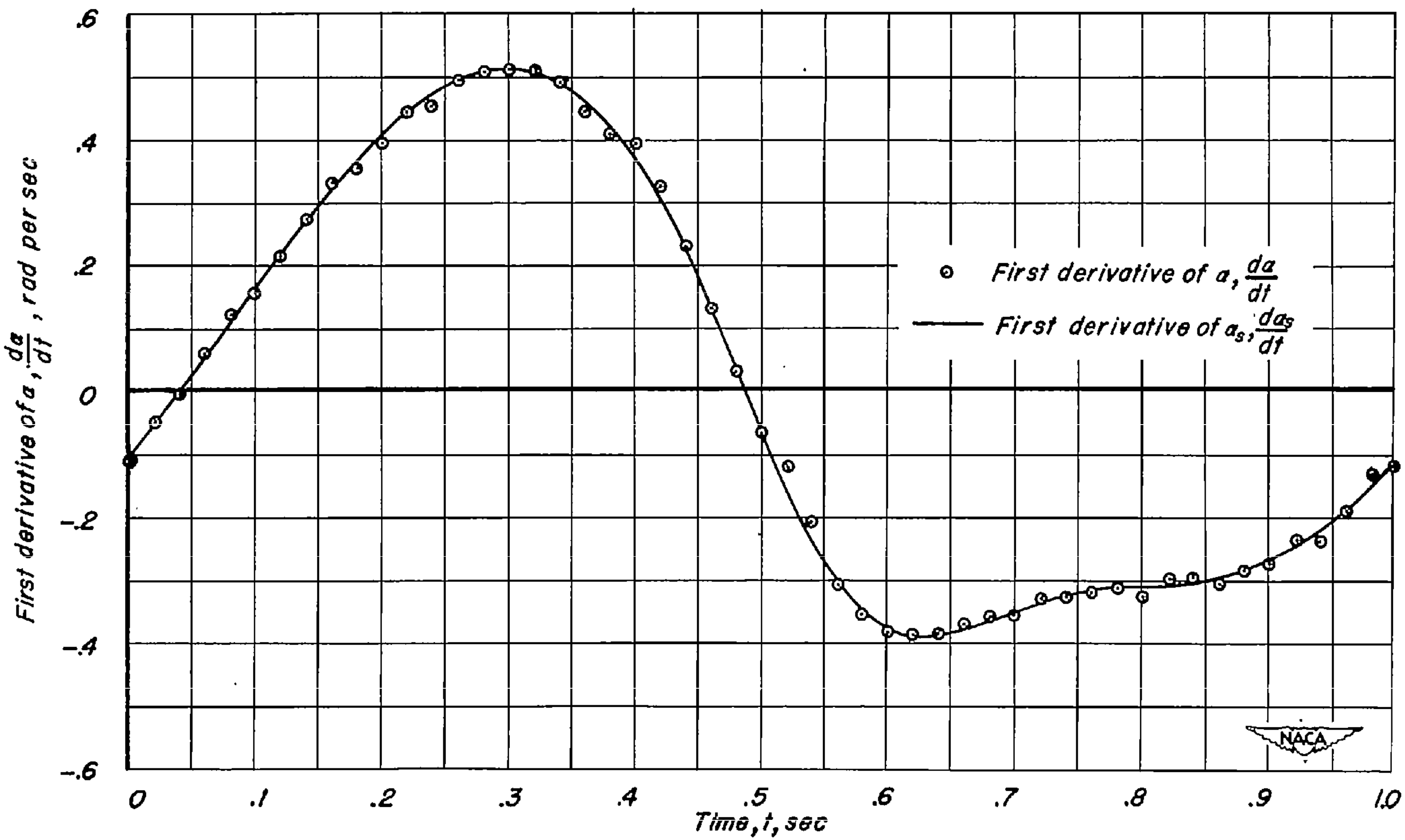



Figure 6.- Comparison of curves for the time rate of change of angle of attack computed from smoothed and unsmoothed data.

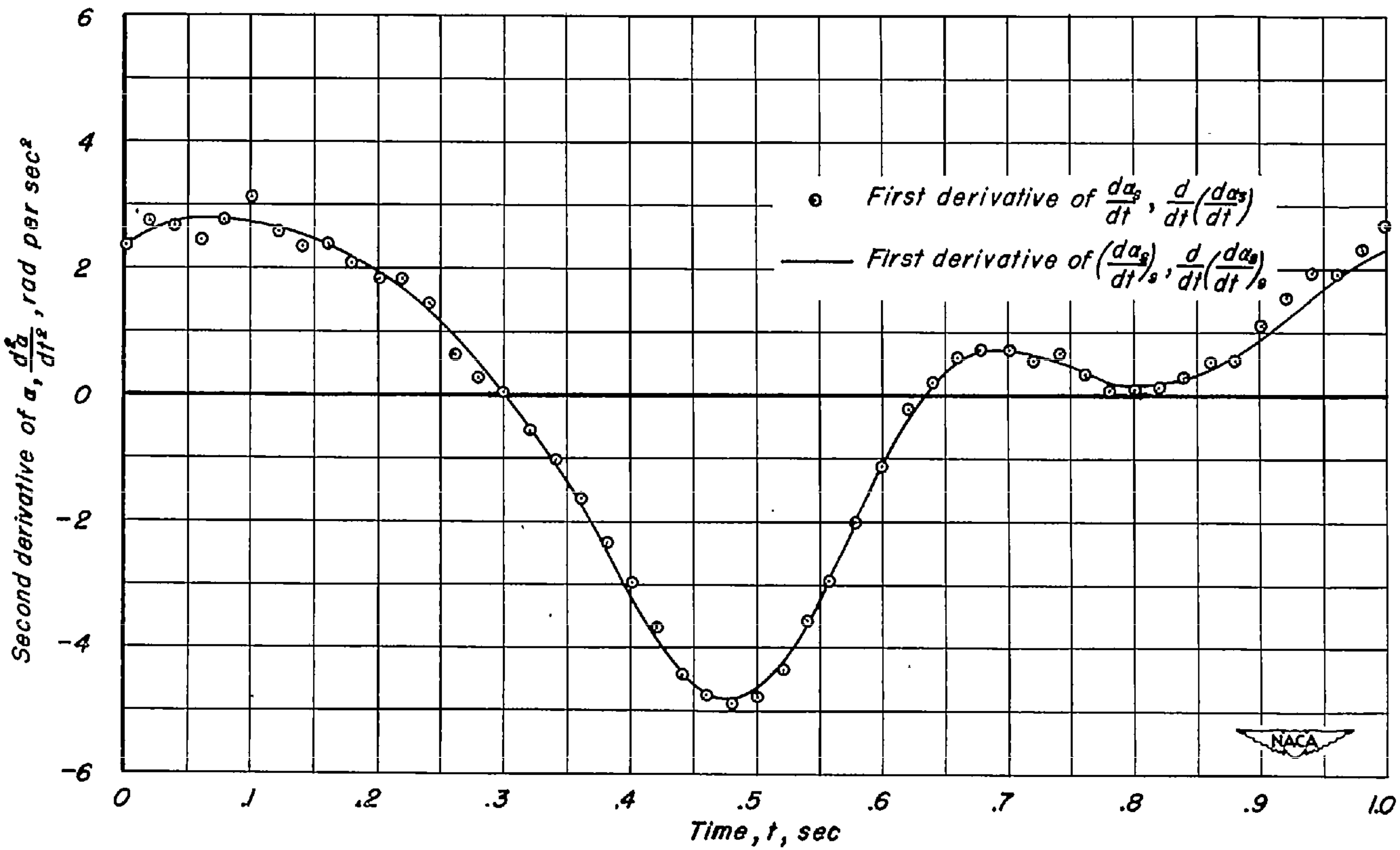



Figure 7.– Comparison of curves for second derivative of angle of attack with respect to time computed from smoothed and unsmoothed data.

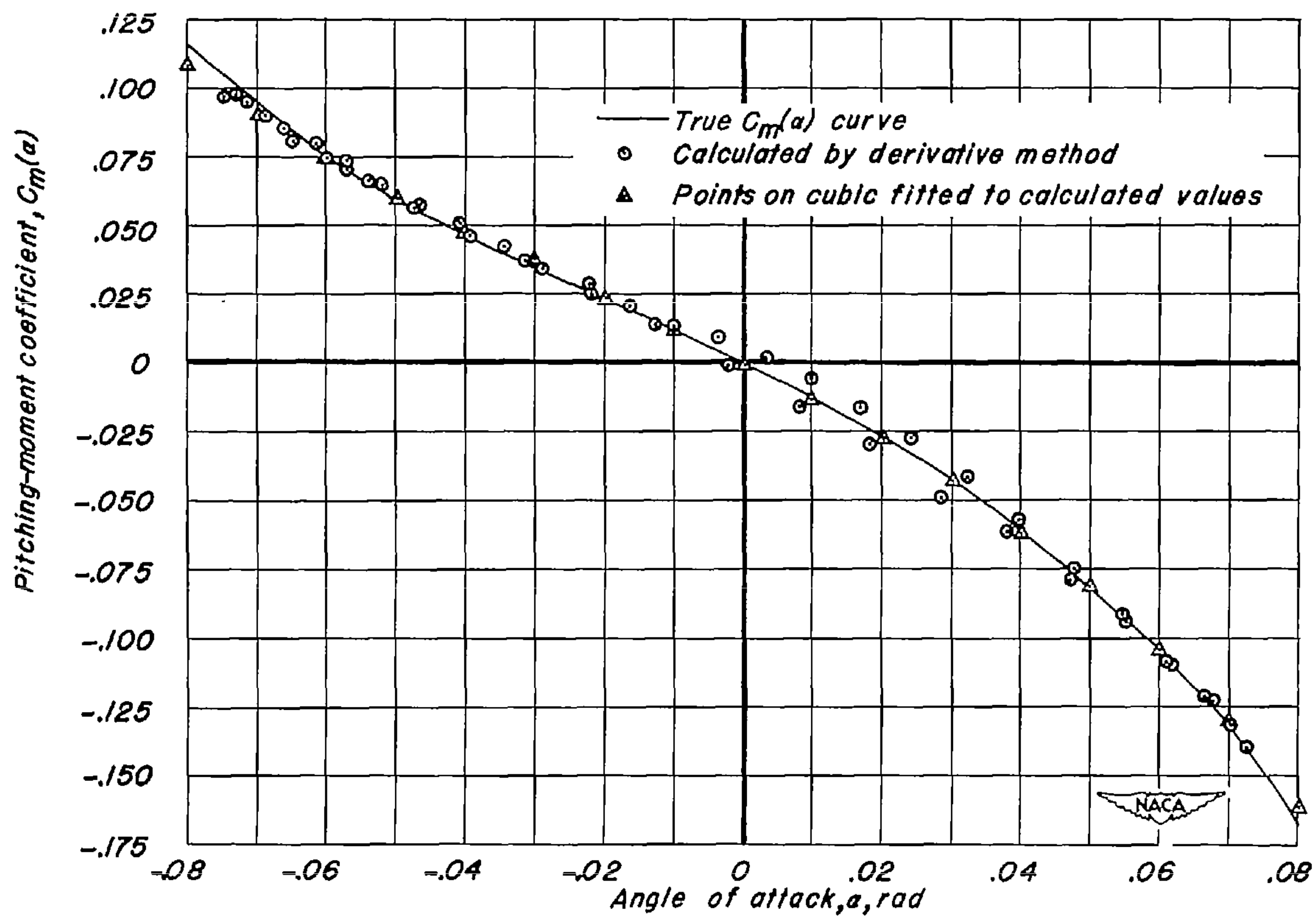



Figure 8.- Comparison of true and calculated nonlinear variations of pitching-moment coefficient with angle of attack.

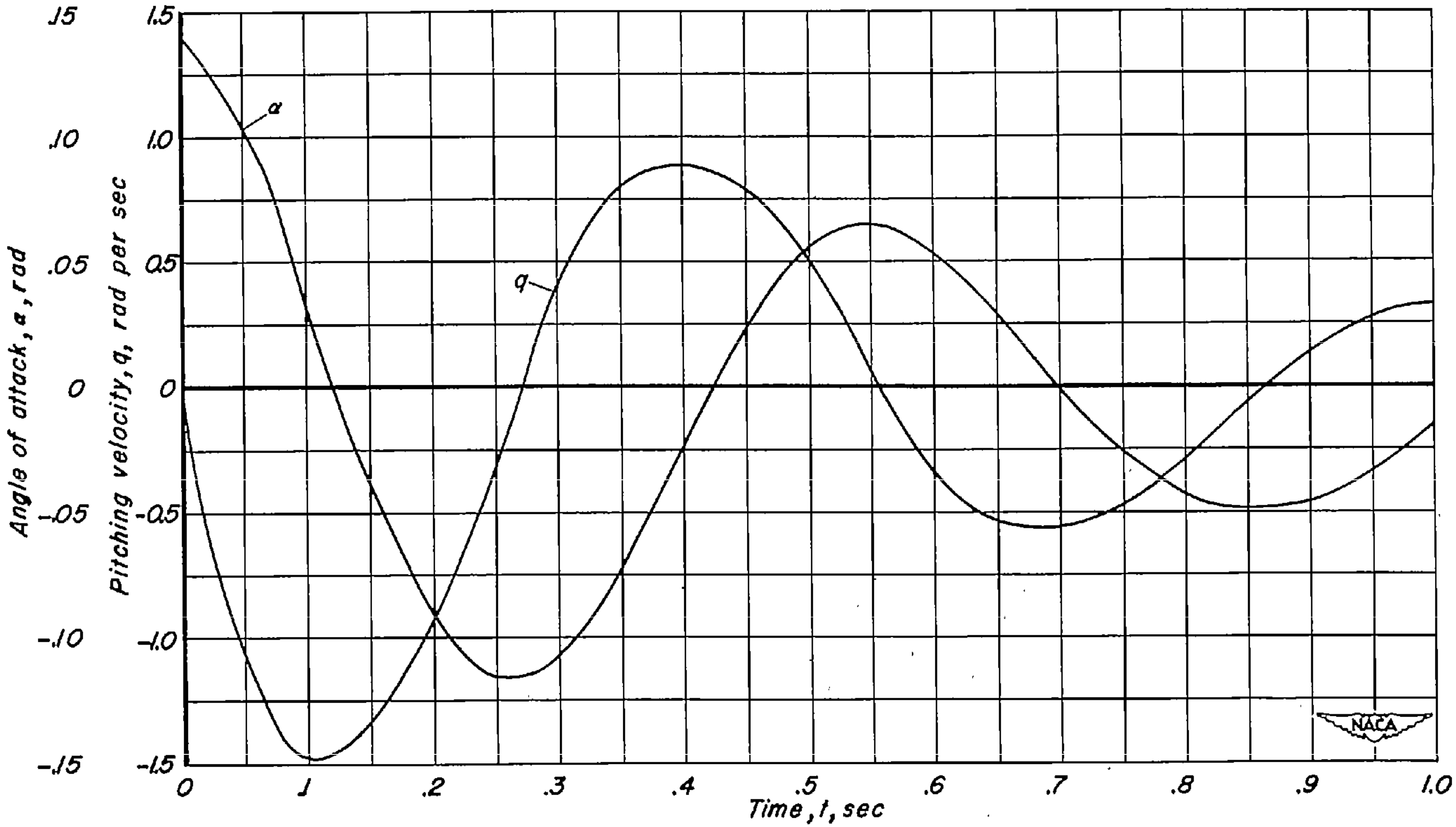



Figure 9.– Transient responses of a missile in angle of attack and pitching velocity.

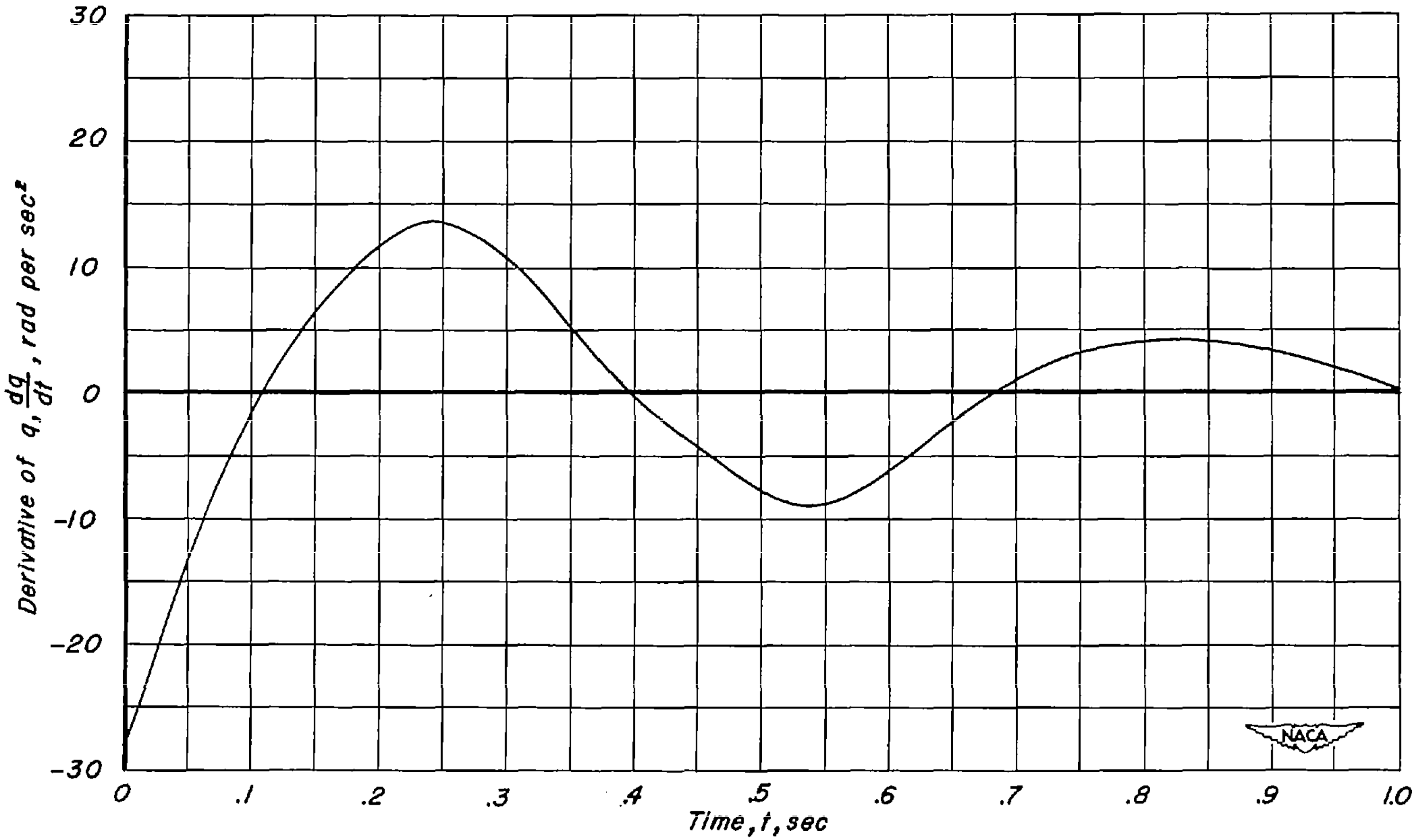



Figure 10.– Numerically computed derivative of the smoothed transient pitching velocity of a missile.

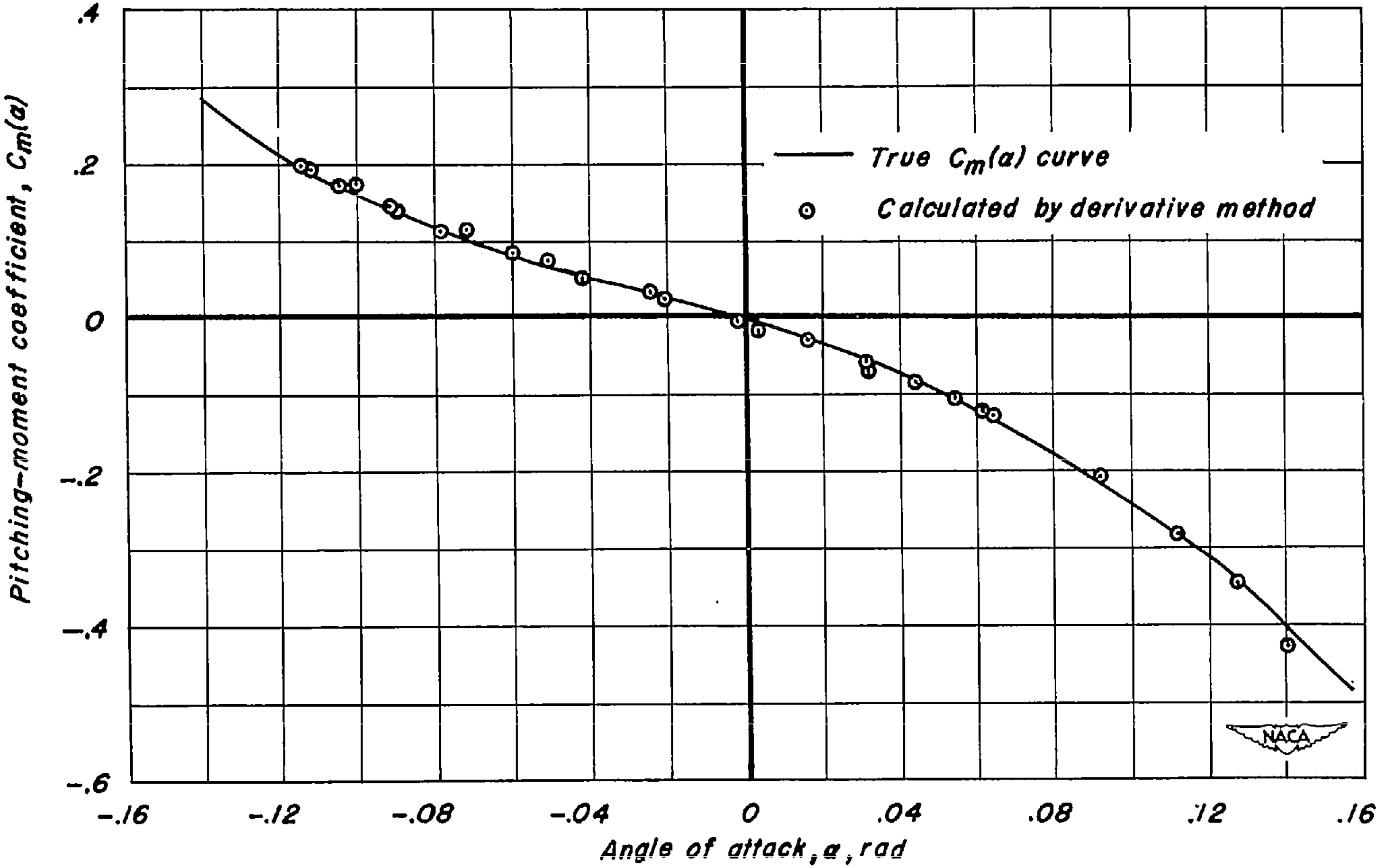



Figure 11.- Comparison of true nonlinear pitching-moment coefficient curves with curve calculated by the derivative method.

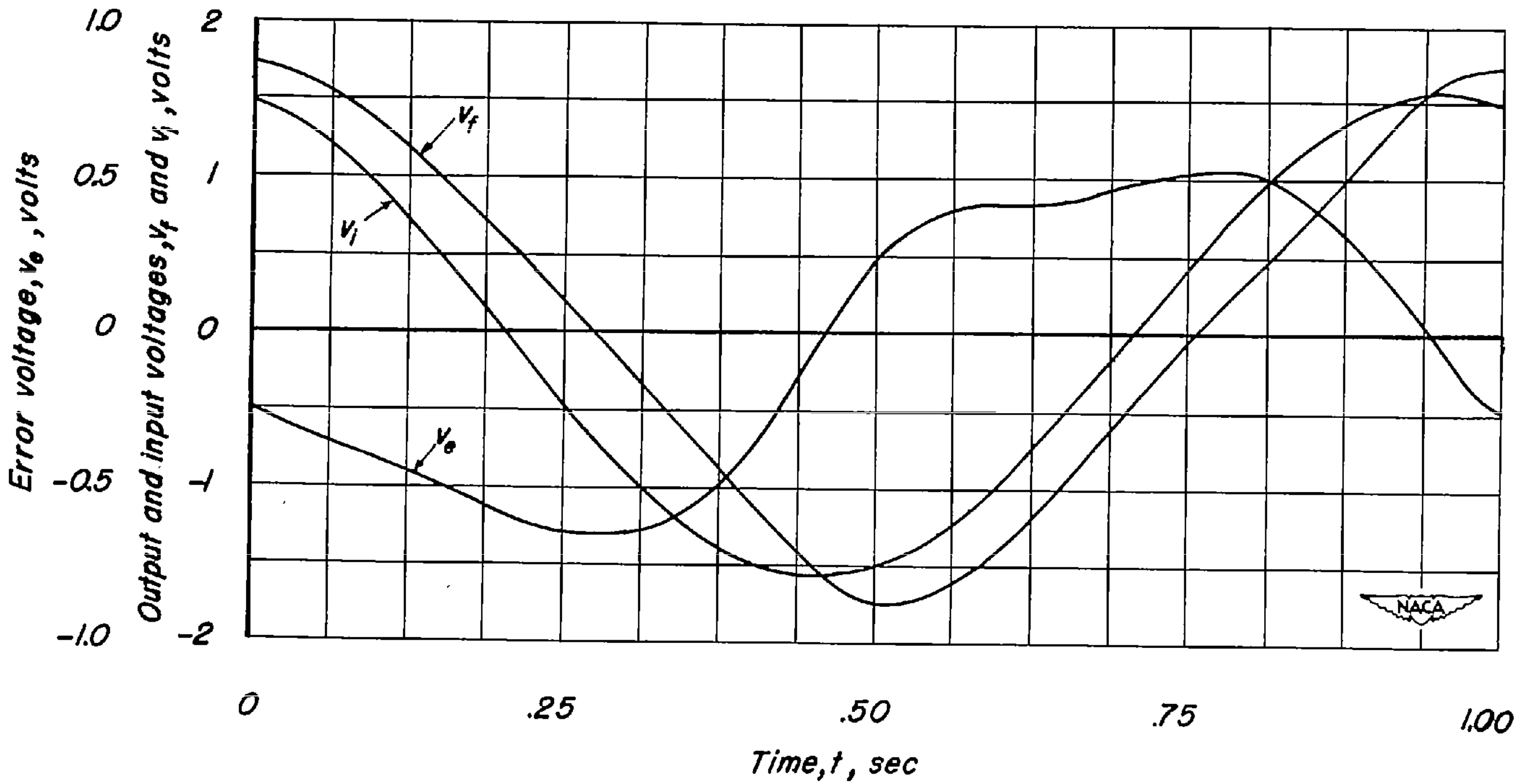
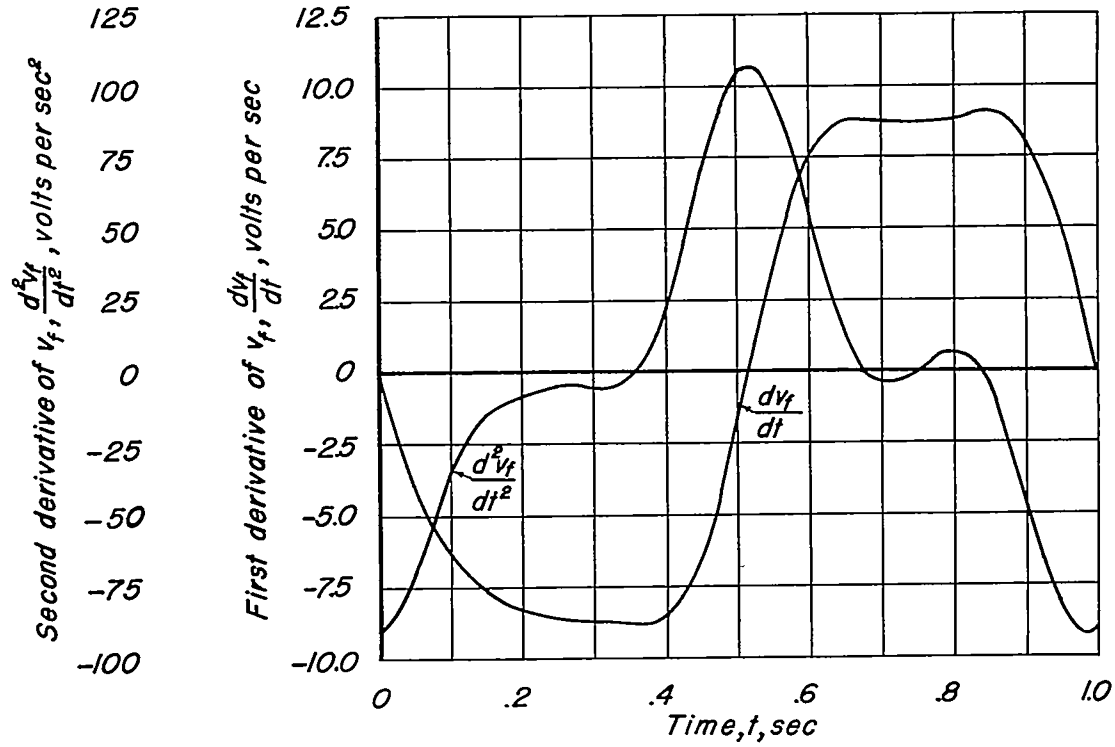
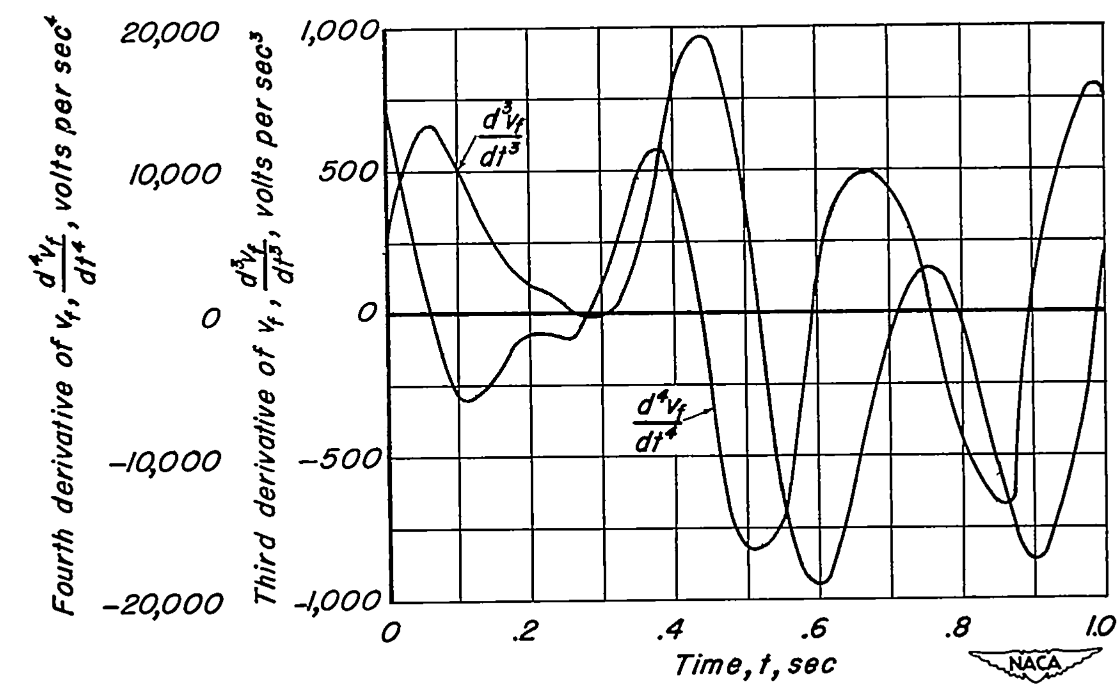



Figure 12.– Output, input, and error voltage of a fourth-order servo system.




(a) First and second derivatives




(b) Third and fourth derivatives

Figure 13.- The first four derivatives with respect to time of the output voltage of a fourth-order servo system.

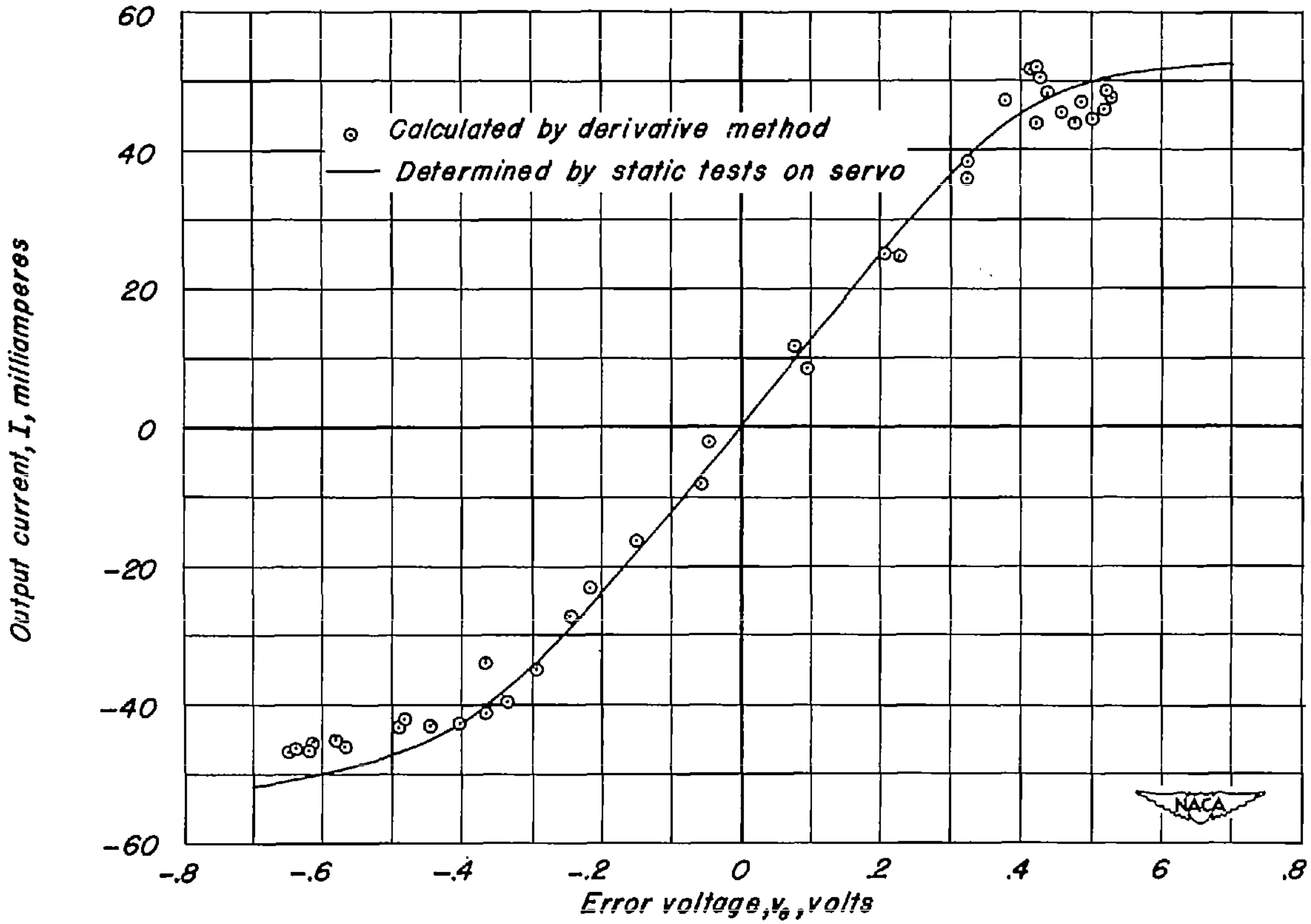



Figure 14.- Comparison of the calculated and measured nonlinear amplifier characteristic curves for a fourth-order servo system.

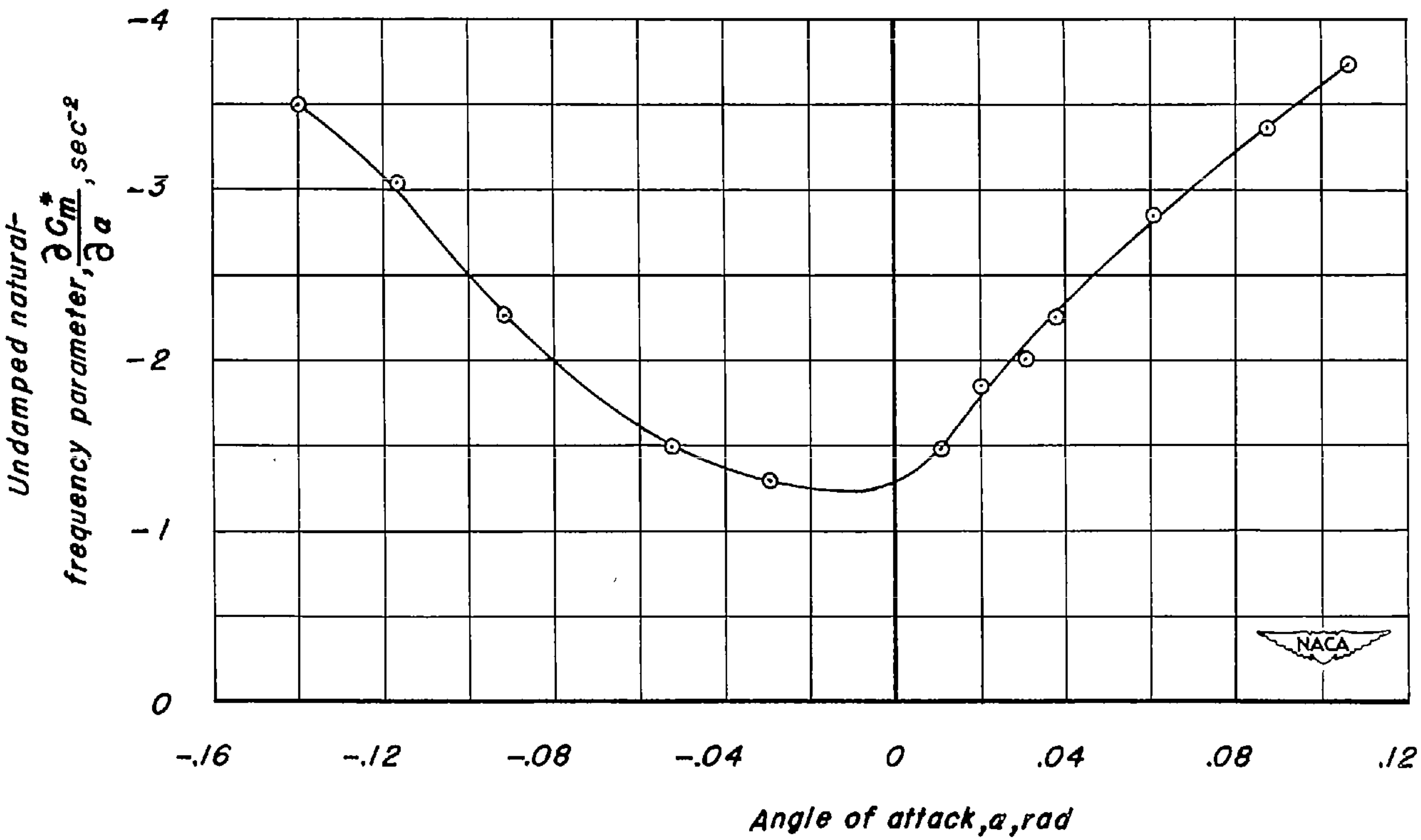



Figure 15.- Variation of the undamped natural-frequency parameter with angle of attack.

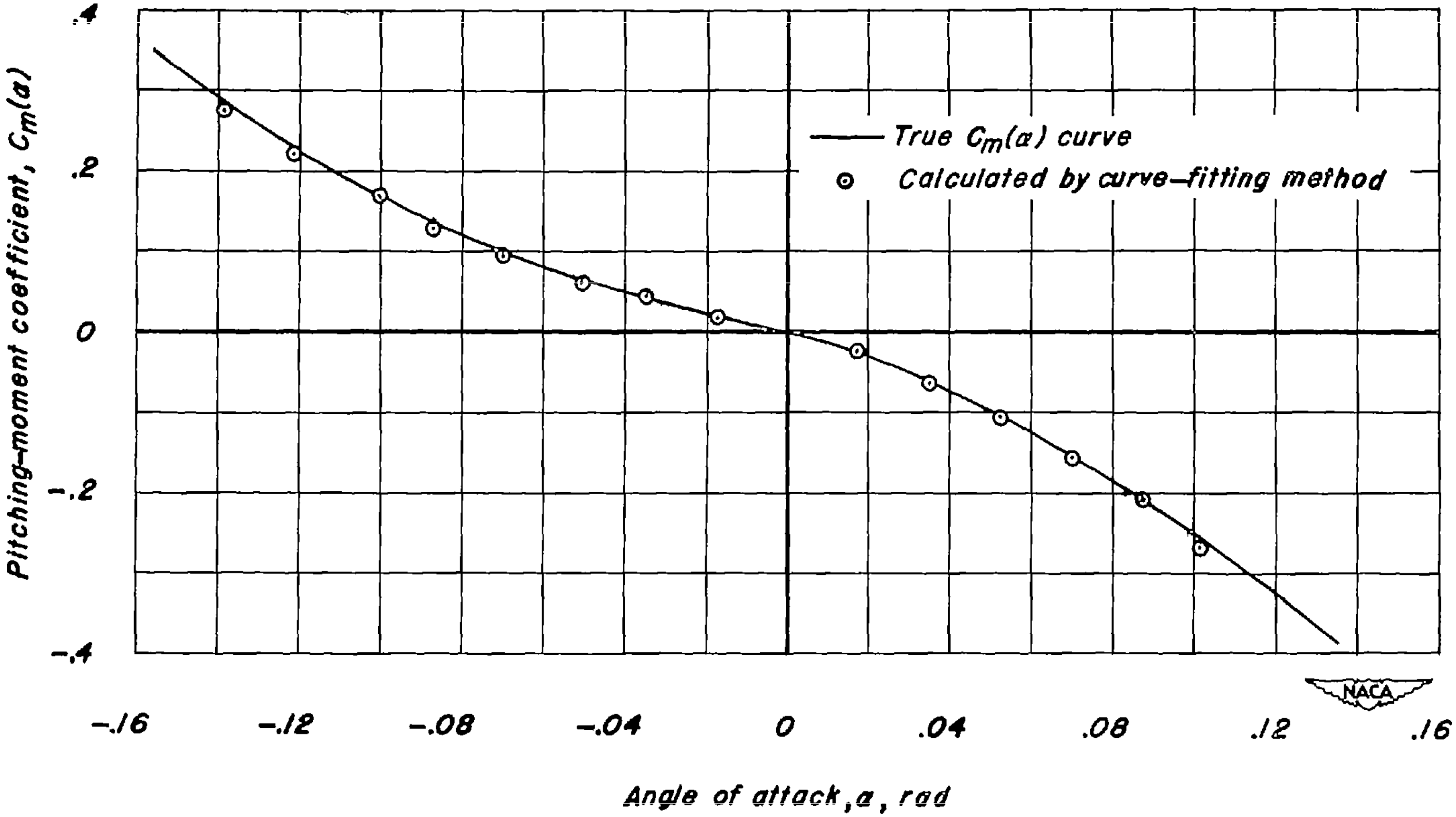



Figure 16.– Comparison of true nonlinear pitching-moment-coefficient curve with curve calculated using the curve-fitting method.

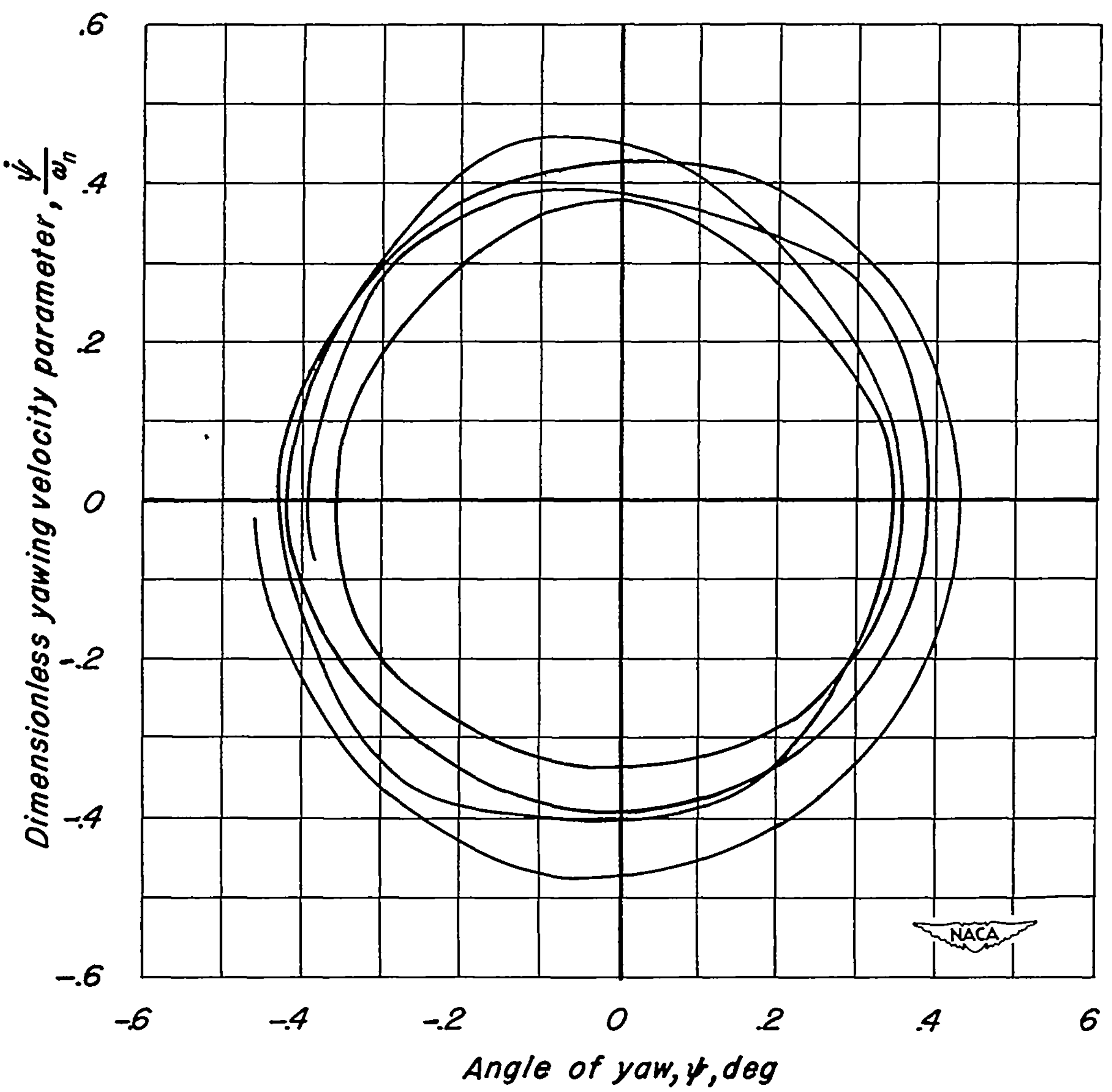



Figure 17.– Phase-plane plot of the self-sustained oscillation of an aircraft model in a wind tunnel.

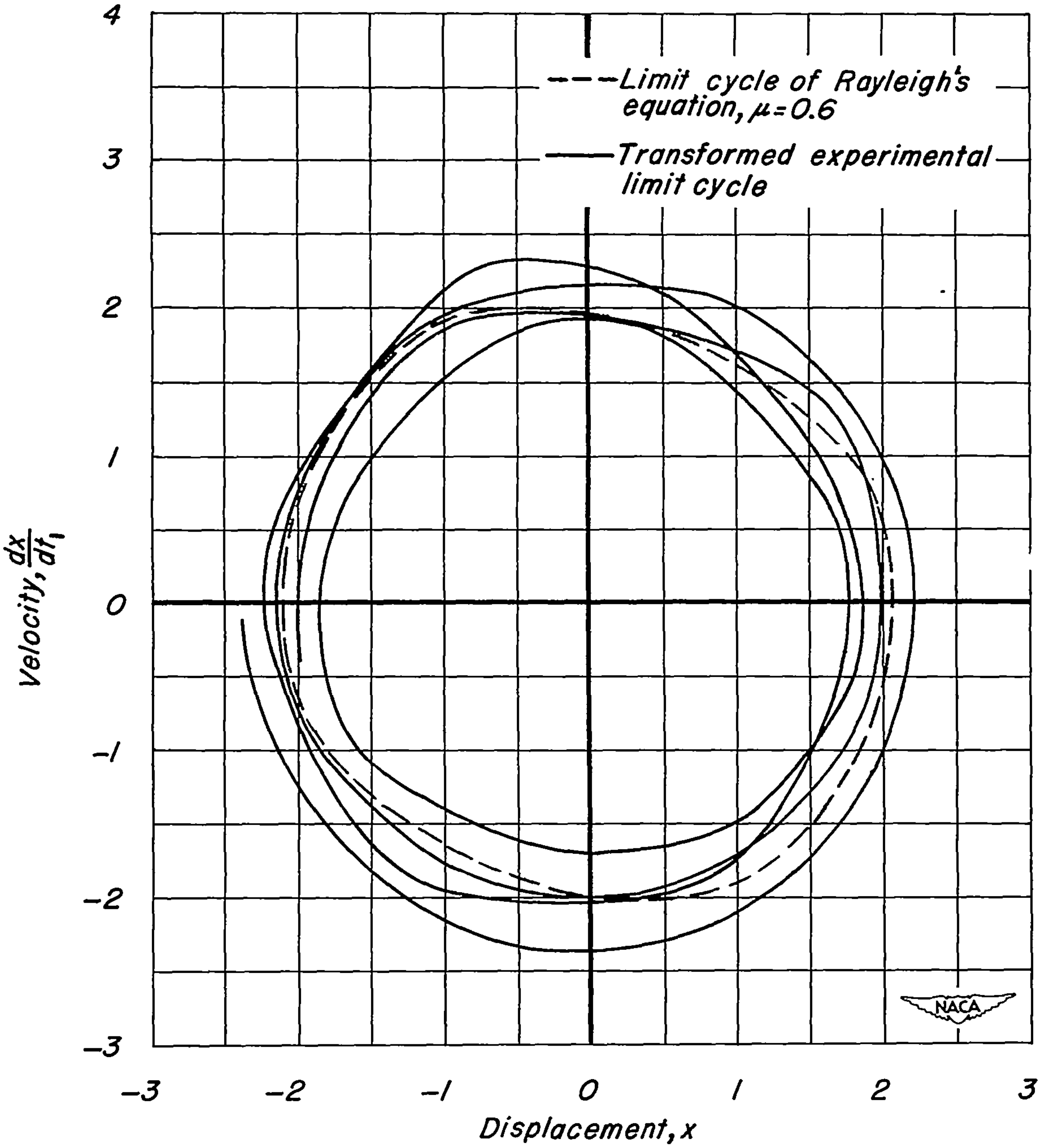



Figure 18.- Comparison of the phase-plane plot of the self-sustained oscillation of an aircraft model and the limit cycle of Rayleigh's equation with $\mu = 0.6$.

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